Solutions Day 33, R 3/21/2024

Topic 15: Inverse, determinant, transpose

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Problem 1. *Let* $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. *Find* A^{-1} *both by the formula and by row reduction.*

Solution: Using the formula: $\det A = 8 - 3 = 5$. So, $A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2/5 & -3/5 \\ -1/5 & 4/5 \end{bmatrix}$

Using row reduction:

$$\left[\begin{array}{cc|cc} 4 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap rows}} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 4 & 3 & 1 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - 4R_1} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -5 & 1 & -4 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{5}R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/5 & 4/5 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2/5 & -3/5 \\ 0 & 1 & -1/5 & 4/5 \end{array}\right]$$

(Same answer with both methods!)

Problem 2. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

(a) *Compute* $\det A$ *by first simplifying* A *using row reduction.*

Solution: There are many ways to do this. Our goal is to get a column with lots of zeros.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow[\text{det. unchanged}]{\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

So, expanding along the first column, $\det A = 1 \cdot \begin{vmatrix} -2 & -1 \\ -1 & 0 \end{vmatrix} = \boxed{-1}$.

(b) *What is Null(A)? (No computation necessary after Part (a).)*

Solution: Since $\det A \neq 0$, Null(A) is trivial.

(c) **Do this after problems 3,4**. *Use row reduction to find* A^{-1}.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{\text{swap } R_2 \text{ and } R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & -2 & -1 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 = -R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -2 & -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 = R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 & -2 \end{array}\right] \xrightarrow{R_3 = -R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 3 & -6 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right] \xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & -4 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix}$. Check: $A{\cdot}A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & -4 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ■

Problem 3. *Compute the determinant of each of the following.*

(a) $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

Solution: Diagonal: $\det A = 3 \cdot 4 = 12$.

(b) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Solution: Diagonal: $\det B = 1 \cdot 4 \cdot (-2) = -8$.

(c) $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

Solution: Trick question: C is not square, so it doesn't have a determinant.

Problem 4. *Compute the determinant:*

(a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

(b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}$.

(c) $C = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 1 & -2 & 0 \\ 3 & 2 & 5 & 2 \end{bmatrix}$

Solution: These are all triangular matrices, so the determinant = product of diagonal entries.

(a) $\det A = 3$, **(b)** $\det B = 25$, **(c)** $\det C = -60$.

Problem 5. *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(a) *Compute* $\det A$.

Solution: First we simplify using row reduction.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[\text{det. unchanged}]{\substack{R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 7R_1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow[\text{det. unchanged}]{R_3 = R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

The row of all zeros implies $\det A = 0$.

(b) *Find Null(A).*

Solution: Since $\det A = 0$, we know Null(A) is nontrivial.

To find Null(A), we continue the row reduction to RREF

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = -\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

To find a null vector, we set the free variable to 1 and solve for the pivot variables. We can do this by inspection by putting our calculation below the matrix as shown.

$$\begin{array}{ccc} \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} \text{P} & \text{P} & \text{F} \\ x_1 & x_2 & x_3 \\ 1 & -2 & 1 \end{array} \end{array}$$

So $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for Null(A) and $\left\{ c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ equals Null(A).

(c) *Use row reduction to find* A^{-1}

Solution: Since $\det A = 0$, A is singular, i.e., A^{-1} does not exist.