Day 34, F 3/22/2024

Topic 16: Eigenstuff (day 1)

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1 Agenda

- Eigenvalues and eigenvectors ***** More important than $x' = ax$ *****
  - Definition, algorithm, 2×2 shortcuts
  - Complex λ (in problems)
  - Repeated λ (in problems, if time)
  - Diagonal matrices
- Tomorrow: Systems of DEs, decoupling, diagonalization

2 Topic 15 key point

A a square matrix.

$$\det A = 0 \longleftrightarrow \text{Null(A) is nontrivial (also, no } A^{-1}).$$

Reason: For a square matrix, $\det A = 0 \longleftrightarrow$ RREF has a row of zeros.

3 Definition of eigenvalues/eigenvectors

Eigen = own, characteristic.

$A = n \times n$ matrix $\longleftarrow$ Important that A is square.

Definition: If $\mathbf{v}$ is a nonzero vector, λ is a scalar and

$$A\mathbf{v} = \lambda\mathbf{v} \quad \text{*** Definition –keep it in mind.}$$

then λ is an eigenvalue of A and $\mathbf{v}$ is a corresponding eigenvector.

We call $A\mathbf{v} = \lambda\mathbf{v}$ the eigenequation. Please remember this. It is the answer to the question: What is an eigenvalue/vector? Don't lose sight of it as we learn computational algorithms.

4 Computational algorithm

- Model example
- Justification
- 2×2 shortcuts
- Complex and repeated eigenvalues (in problems)

Example 1. (Model example) Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Find its eigenvalues and a basis for each eigenspace.

Solution: Step 1. Find the eigenvalues λ: $|A - \lambda I| = 0$ (**characteristic equation)**

$$A - \lambda I = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{bmatrix}.$$

Taking the determinant and setting it to 0 gives

$$\det(A - \lambda I) = (4-\lambda)(2-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = 0.$$

The roots of this are $\boxed{\lambda = 5,\ 1.}$

Step 2. For each eigenvalue, find basis vectors for the eigenspace, i.e., find a basis of $\text{Null}(A - \lambda I)$.

$\lambda_1 = 5$: $A - \lambda I = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$. This has RREF $R = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$. The null space is 1 dimensional, a basis is $\mathbf{v_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$\lambda_1 = 1$: $A - \lambda I = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$. This has RREF $R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The null space is 1 dimensional, a basis is $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Remember, any scalar multiple of these eigenvectors is also an eigenvector with the same eigenvalue.

Summary:	eigenvalues	5,	1
	basic eigenvectors	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$,	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Let's reemphasize a key point: Eigenspaces are null spaces.

4.1 Justification for the algorithm

Eigenequation: $A\mathbf{v} = \lambda \mathbf{v} = \lambda I \mathbf{v}$ (Need $\mathbf{v} \neq 0$, λ a scalar)
$\longrightarrow A\mathbf{v} - \lambda I \mathbf{v} = 0 \longrightarrow (A - \lambda I)\mathbf{v} = 0.$

This says that $\mathbf{v}$ is in $\text{Null}(A - \lambda I)$.

Since $\mathbf{v} \neq 0$, $\text{Null}(A - \lambda I)$ must be nontrivial.

This can only happen if $\det(A - \lambda I) = 0$, i.e., if λ satisfies the characteristic equation.

In short, $\boxed{\text{eigenvalues } \lambda \text{ have } \det(A - \lambda I) = 0, \text{ eigenvectors } \mathbf{v} \text{ are in Null}(A - \lambda I).}$

4.2 Shortcut for 2 by 2 matrices

For 2×2 matrices, we don't really need to use row reduction to find eigenspaces.

In our model example with $\lambda = 5$, we had $A - \lambda I = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$. We needed a null vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ This must be perpendicular to the top row $\begin{bmatrix} -1 & 3 \end{bmatrix}$. So we can take

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(We switched the order of the entries of the top row and changed one sign.)

Note: Since we know the null space is nontrivial, this must be perpendicular to the bottom row as well. (Check this!)

5 Diagonal matrices (nice and easy!)

Example 2. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Verify the eigenpairs are

$$\begin{array}{rcllll} \lambda & = & 1, & 3, & 5, & \text{(diagonal entries)} \\ \mathbf{v} & = & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \text{(standard basis)} \end{array}$$

Solution: Check the eigenequation

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad \text{So, } A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \checkmark$$

$$A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}. \quad \text{So, } A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \checkmark$$

$$A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}. \quad \text{So, } A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 5 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \checkmark$$