Solutions Day 34, F 3/22/2024

Topic 16: Eigenstuff (day 1)

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Problem 1. *(Eigenstuff and linearity) Say A is a 3×3 matrix with eigenvalues $\lambda = 3,\ 4,\ 5$ and corresponding eigenvectors* $\begin{bmatrix}1\\2\\1\end{bmatrix},\ \begin{bmatrix}1\\0\\1\end{bmatrix},\ \begin{bmatrix}0\\0\\1\end{bmatrix}$.

Compute $A\begin{bmatrix}1\\2\\1\end{bmatrix},\quad A\left(\begin{bmatrix}1\\2\\1\end{bmatrix}+\begin{bmatrix}1\\0\\1\end{bmatrix}\right),\quad A\begin{bmatrix}0\\2\\0\end{bmatrix}$.

Solution: $A\begin{bmatrix}1\\2\\1\end{bmatrix} = 3\cdot\begin{bmatrix}1\\2\\1\end{bmatrix}$ by the definition of eigenvalue and eigenvector.

Likewise $A\left(\begin{bmatrix}1\\2\\1\end{bmatrix}+\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = A\begin{bmatrix}1\\2\\1\end{bmatrix} + A\begin{bmatrix}1\\0\\1\end{bmatrix} = 3\begin{bmatrix}1\\2\\1\end{bmatrix} + 4\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}7\\6\\7\end{bmatrix}$.

By inspection we see, $\begin{bmatrix}0\\2\\0\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix}$. So,

$$A\begin{bmatrix}0\\2\\0\end{bmatrix} = A\begin{bmatrix}1\\2\\1\end{bmatrix} - A\begin{bmatrix}1\\0\\1\end{bmatrix} = 3\begin{bmatrix}1\\2\\1\end{bmatrix} - 4\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\6\\-1\end{bmatrix}.$$

Problem 2. *Find the eigenvalues and a basis for each eigenspace of the given matrices. Check your answer by multiplying the matrix times the vector.*

$$A = \begin{bmatrix}6 & 5\\1 & 2\end{bmatrix} \qquad B = \begin{bmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 6\end{bmatrix}.$$

Solution: A: characteristic equation: $|A - \lambda I| = \begin{vmatrix}6-\lambda & 5\\1 & 2-\lambda\end{vmatrix} = \lambda^2 - 8\lambda + 7 = 0$.

Roots: 1, 7 (eigenvalues)

Eigenspaces:

$\lambda = 1$: $A - \lambda I = \begin{bmatrix}5 & 5\\1 & 1\end{bmatrix} \quad \Rightarrow \overbrace{\text{basis for egenspace}}^{\text{basis for Null}(A-I)}$ is $\begin{bmatrix}-5\\5\end{bmatrix}$ or better $\begin{bmatrix}-1\\1\end{bmatrix}$.

$\lambda = 7$: $A - \lambda I = \begin{bmatrix}-1 & 5\\1 & -5\end{bmatrix} \quad \Rightarrow \overbrace{\text{basis for egenspace}}^{\text{basis for Null}(A-7I)}$ is $\begin{bmatrix}5\\1\end{bmatrix}$.

Check: $A\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}6 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix} = 1\cdot\begin{bmatrix}-1\\1\end{bmatrix}$ ■

$$A\begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}6 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}35\\7\end{bmatrix} = 7\cdot\begin{bmatrix}5\\1\end{bmatrix} \qquad \blacksquare$$

Summary: eigenvalue/eigenvector pairs for A are $\boxed{\begin{matrix}\lambda & = 1, & 7\\ \mathbf{v} & = \begin{bmatrix}-1\\1\end{bmatrix} & \begin{bmatrix}5\\1\end{bmatrix}\end{matrix}}$.

B: characteristic equation $|B-\lambda I| = \begin{vmatrix}1-\lambda & 2 & 3\\ 0 & 4-\lambda & 5\\ 0 & 0 & 6-\lambda\end{vmatrix} = (1-\lambda)(4-\lambda)(6-\lambda) = 0.$

Eigenvalues (roots): $\lambda = 1,\ 4,\ 6$. (B is triangular, so eigenvalues = diagonal entries.)

Eigenspaces:

$\lambda = 1$: $B-\lambda I = \begin{bmatrix}0 & 2 & 3\\ 0 & 3 & 5\\ 0 & 0 & 5\end{bmatrix}$. We need a basis of Null$(B-I)$. Using row reduction to find the RREF we get

$$\begin{matrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\\ \begin{matrix}\text{F} & \text{P} & \text{P}\\ 1 & 0 & 0\end{matrix}\end{matrix} \Rightarrow \text{basis of eigenspace} = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \text{(eigenvector associated with } \lambda = 1)$$

$\lambda = 4$: $B-\lambda I = \begin{bmatrix}-3 & 2 & 3\\ 0 & 0 & 5\\ 0 & 0 & 2\end{bmatrix}$. Find the null space by row reduction:

$$\text{RREF} = \begin{matrix}\begin{bmatrix}1 & -2/3 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\\ \begin{matrix}\text{P} & \text{F} & \text{P}\\ 2/3 & 1 & 0\end{matrix}\end{matrix} \Rightarrow \text{basic eigenvector} = \begin{bmatrix}2/3\\1\\0\end{bmatrix} \text{ or } \begin{bmatrix}2\\3\\0\end{bmatrix}$$

$\lambda = 6$: $B-\lambda I = \begin{bmatrix}-5 & 2 & 3\\ 0 & -2 & 5\\ 0 & 0 & 0\end{bmatrix}$. Find the null space by row reduction.

$$\text{RREF} = \begin{matrix}\begin{bmatrix}1 & 0 & -8/5\\ 0 & 1 & -5/2\\ 0 & 0 & 0\end{bmatrix}\\ \begin{matrix}\text{P} & \text{P} & \text{F}\\ 8/5 & 5/2 & 1\end{matrix}\end{matrix} \Rightarrow \text{basic eigenvector} = \begin{bmatrix}8/5\\5/2\\1\end{bmatrix}.$$

Summary: $\boxed{\begin{matrix}\lambda & = 1, & 4, & 6\\ \mathbf{v} & = \begin{bmatrix}1\\0\\0\end{bmatrix}, & \begin{bmatrix}2\\3\\0\end{bmatrix}, & \begin{bmatrix}8/5\\5/2\\1\end{bmatrix}\end{matrix}}$.

Check:

$$B\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \blacksquare$$

$$B\begin{bmatrix}2\\3\\0\end{bmatrix} = \begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}\begin{bmatrix}2\\3\\0\end{bmatrix} = \begin{bmatrix}8\\12\\0\end{bmatrix} = 4\begin{bmatrix}8\\12\\0\end{bmatrix} \qquad \blacksquare$$

$$B\begin{bmatrix}8/5\\5/2\\1\end{bmatrix} = \begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}\begin{bmatrix}8/5\\5/2\\1\end{bmatrix} = \begin{bmatrix}48/5\\15\\6\end{bmatrix} = 6\begin{bmatrix}8/5\\5/2\\1\end{bmatrix} \qquad \blacksquare$$

Problem 3. *(Complex λ)* $A = \begin{bmatrix}3&5\\-5&3\end{bmatrix}$ *–same instructions as Problem 2.*

Solution: Characteristic equation: $|A-\lambda I| = \begin{vmatrix}3-\lambda & 5\\ -5 & 3-\lambda\end{vmatrix} = (3-\lambda)^2 + 25 = 0.$

Eigenvalues (roots): $(3-\lambda)^2 = -25 \quad \Rightarrow 3-\lambda = \pm 5i \quad \Rightarrow \lambda = 3 \pm 5i.$

Eigenspaces

$\lambda = 3+5i$: $\quad A-\lambda I = \begin{bmatrix}-5i & 5\\ -5 & -5i\end{bmatrix} \xrightarrow{2\times 2 \text{ trick}}$ basic eigenvector $= \begin{bmatrix}5\\5i\end{bmatrix}$ or $\begin{bmatrix}1\\i\end{bmatrix}$.

$\lambda = 3-5i$: $\quad$ Eigenvector = conjugate of one for $3+5i$: $\begin{bmatrix}1\\-i\end{bmatrix}$.

Summary of eigenvalue/vector pairs:

λ	$=$	$3+5i$	$3-5i$
$\mathbf{v}$	$=$	$\begin{bmatrix}1\\i\end{bmatrix}$,	$\begin{bmatrix}1\\-i\end{bmatrix}$

Check: $\quad A\begin{bmatrix}1\\i\end{bmatrix} = \begin{bmatrix}3&5\\-5&3\end{bmatrix}\begin{bmatrix}1\\i\end{bmatrix} = \begin{bmatrix}3+5i\\-5+3i\end{bmatrix} = (3+5i)\cdot\begin{bmatrix}1\\i\end{bmatrix} \quad \blacksquare$

(The other pair is similar.)

Problem 4. *(Diagonal matrices) Same instructions as Problem 2.*

$$A = \begin{bmatrix}2&0\\0&3\end{bmatrix}, \qquad B = \begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}.$$

Solution: Eigenvalues are diagonal entries. Eigenvectors are corresponing standard basis vectors.

For A:

λ	$=$	2,	3
$\mathbf{v}$	$=$	$\begin{bmatrix}1\\0\end{bmatrix}$,	$\begin{bmatrix}0\\1\end{bmatrix}$

For B:

λ	$=$	2,	3,	4
$\mathbf{v}$	$=$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$,	$\begin{bmatrix}0\\1\\0\end{bmatrix}$,	$\begin{bmatrix}0\\0\\1\end{bmatrix}$

Checking this is easy.

Problem 5. *(Repeated λ) Same instructions as Problem 2.*

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Why is B called 'defective'?

Solution: A is diagonal, so $\boxed{\begin{array}{ll} \lambda & = 2, \quad 2 \\ \mathbf{v} & = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array}}$

B is triangular, so eigenvalues = diagonal entries: 2, 2.

Eigenspace for $\lambda = 2$.

$$B - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{basis of Null}(B-2I) \text{ is } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ (one dimensional).}$$

$$\begin{array}{cc} \text{F} & \text{P} \\ 1 & 0 \end{array}$$

Summary $\boxed{\begin{array}{lll} \lambda & = & 2, 2 \\ \mathbf{v} & = & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}}$ (2 eigenvalues, only 1 independed eigenvector.)

This is called defective because there aren't the expected number of independent eigenvectors.

C is triangular, so eigenvalues = diagonal entries: 2, 3, 2.

Eigenspaces:

$$\lambda = 2: \quad C - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{swap } R_1, R_2} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Basic eigenvectors: } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

$$\begin{array}{ccc} \text{F} & \text{P} & \text{F} \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{array}$$

$$\lambda = 3: \quad C - 3I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Basic eigenvector: } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\begin{array}{ccc} \text{P} & \text{F} & \text{P} \\ 0 & 1 & 0 \end{array}$$

Summary $\boxed{\begin{array}{lll} \lambda & = & 2, 2, \quad 3 \\ \mathbf{v} & = & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{array}}$.