Day 35, M 4/1/2024

Topic 16: Eigenstuff (day 2)
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1 Agenda

- Review and finish last class
  - Definition of eigenvalues/eigenvectors
  - Computational algorithm ***
  - Problems (real and complex λ)
  - Diagonal matrices (are easy)
  - 2×2 shortcuts
- Solving systems of DEs (introduction)
- Diagonalization
- Decoupling

2 Eigenvalues/eigenvectors definition

A is an $n \times n$ matrix (square)

If $A\mathbf{v} = \lambda\mathbf{v}$ for a nontrivial $\mathbf{v}$, then λ is an eigenvalue of A and $\mathbf{v}$ is a corresponding eigenvalue.

– This is the answer to: "What is an eigenvalue/eigenvector?"

Example 1. Suppose A has eigenvalues $3,\ -2$ with corresponding eigenvectors $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$.

Compute $A\left(\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}\right)$.

Solution: This equals $\quad 3\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ -9 \\ -5 \\ -7 \end{bmatrix}$.

Reformulation: $A\mathbf{v} = \lambda\mathbf{v} \longleftrightarrow (A - \lambda I)\mathbf{v} = 0$.

That is, $\text{Null}(A - \lambda I)$ is nontrivial $\ \det(A - \lambda I) = 0, \quad \mathbf{v} \in \text{Null}(A - \lambda I)$.

3 Computational algorithm (outline)

Characteristic equation: $\quad \det(A - \lambda I) = 0. \quad$ Roots = eigenvalues.

Eigenspaces = $\text{Null}(A - \lambda I)$. Basic eigenvectors = bases of eigenspaces.

Do problems 1-5 from Topic 16 Day 1 problems.

4 Systems of DEs in matrix form

Example 2. Consider the system: $\begin{array}{rcrcr} x' & = & 6x & + & 5y \\ y' & = & x & + & 2y \end{array}$

- x, y are coupled.
- 2 first-order DEs = 2nd order system.
- Matrix form: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- Abstractly: $\mathbf{x}' = A\mathbf{x}$. $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$.
- Homogeneous: $\mathbf{x}' - A\mathbf{x} = 0$ ($\mathbf{x}$ on left, 0 on right).

Key observation: If λ, $\mathbf{v}$ is an eigenpair of A, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.

Reason: Method of optimism. Guess a solution of the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v},\ \lambda \text{ scalar },\ \mathbf{v} \text{ constant vector.}$$

Substitute this into the equation

$$\left.\begin{array}{ccc} \mathbf{x}' & = & A\mathbf{x} \\ \updownarrow & & \updownarrow \\ \lambda \cancel{e^{\lambda t}}\mathbf{v} & = & \cancel{e^{\lambda t}}A\mathbf{v} \end{array}\right\} \longrightarrow \lambda\mathbf{v} = A\mathbf{v} = \text{ eigenequation.}$$

Example 3. Do Problem 1a for today.

5 Diagonalization

Suppose A is $n \times n$ with n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Let $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$ = matrix with eigenvectors as columns.

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \text{diagonal matrix of eigenvalues.}$$

then, in diagonalized form, $A = S\Lambda S^{-1}$. (Proof in Topic 16 notes and below.)

Example 4. $\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 1 & 1 \end{bmatrix}^{-1}$.

5.1 Uses of diagonalization

Powers: $A^n = S\Lambda^n S^{-1}$.

Proof (by example): $A^2 = S\Lambda \overbrace{S^{-1}\cdot S}^{I} \Lambda S^{-1} = S\Lambda^2 S^{-1}$.

Example 5. $\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}^4 = \begin{bmatrix} -1 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}^4 \begin{bmatrix} -1 & 5 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^4 & 0 \\ 0 & 7^4 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 1 & 1 \end{bmatrix}^{-1}.$

Inverses: $A^{-1} = S\Lambda^{-1}S^{-1}$.

Proof: $A\cdot(S\Lambda^{-1}S^{-1}) = S\Lambda \overbrace{S\cdot S^{-1}}^{I} \Lambda^{-1}S^{-1} = S \overbrace{\Lambda\Lambda^{-1}}^{I} S^{-1} = SS^{-1} = I.$ ■

Determinants: $\det A = \det \Lambda =$ product of eigenvalues.

Proof: $\det A = \det S\Lambda S^{-1} = \det(S)\det(\Lambda)\det(S^{-1}) = \det\Lambda$ (since $\det(S)$ and $\det(S^{-1})$ are reciprocals).

Example 6. Do problems 1b, 1c.

6 Decoupling

Example 7. The system $\begin{array}{rcl} u' & = & u \\ v' & = & 7v \end{array}$ is uncoupled $\longrightarrow$ $\begin{array}{l} u = c_1 e^t \\ v = c_2 e^{7t} \end{array}$.

In matrix form: $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$ = a diagonal system.

Coefficient matrix: $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$ has eigenstuff

$$\begin{array}{rcll} \lambda & = & 1, & 7 \\ \mathbf{v} & = & \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array}.$$

So, $\begin{bmatrix} u \\ v \end{bmatrix} = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ = same solution as above.

If $A = S\Lambda S^{-1}$, Λ diagonal, then the system $\mathbf{x}' = A\mathbf{x}$ is decoupled by the change of variables $\mathbf{x} = S\mathbf{u}$ (or $\mathbf{u} = S^{-1}\mathbf{x}$). That is,

$$\underbrace{\mathbf{x}' = A\mathbf{x}}_{\text{coupled}} = S\Lambda S^{-1}\mathbf{x} \quad \Rightarrow \underbrace{S^{-1}\mathbf{x}'}_{\mathbf{u}'} = \Lambda \underbrace{S^{-1}\mathbf{x}}_{\mathbf{u}} \quad \Leftrightarrow \underbrace{\mathbf{u}' = \Lambda\mathbf{u}}_{\text{uncoupled}}$$

Decoupling does not help a lot in 1803, because once we have the eigenstuff, we've already solved the system. We show it because engineers often want to work in decoupled coordinates.

Example 8. Do problems 1d, 2.

7 Proof of diagonalization formula

(Probably won't do in class. Also see the Topic 16 notes.)

Our first step is to write this as $AS = S\Lambda$.

Our strategy is to show these give the same result when multiplied by any standard basis vector.

The key is that S times a standard basis vector is an eigenvector of A.

Let's see this by example, using our usual $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$.

We have eigenpairs: $\lambda_1 = 1$, $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\quad \lambda_2 = 7$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

So, $S = \begin{bmatrix} -1 & 5 \\ 1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$.

Standard basis: $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Note: $S\mathbf{e}_1 = S\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \mathbf{v_1}$. Likewise, $S\mathbf{e}_2 = \mathbf{v}_2$.

Note also: $\Lambda\mathbf{e}_1 = 1 \cdot \mathbf{e}_1$, $\Lambda\mathbf{e}_2 = 7 \cdot \mathbf{e}_2$.

Now,

$$\begin{array}{ll} AS\mathbf{e}_1 = A\mathbf{v}_1 = 1 \cdot \mathbf{v}_1 & S\Lambda\mathbf{e}_1 = S(1 \cdot \mathbf{e}_1) = 1 \cdot \mathbf{v}_1 \quad \text{(the same)} \\ AS\mathbf{e}_2 = A\mathbf{v}_2 = 7 \cdot \mathbf{v}_2 & S\Lambda\mathbf{e}_2 = S(7 \cdot \mathbf{e}_2) = 7 \cdot \mathbf{v}_1 \quad \text{(the same)} \end{array}$$

So AS and $S\Lambda$ give the same result when multiplied times a standard basis vector. By linearity, they give the same result when multiplied times any linear combination of the standard basis vectors, i.e., times any vector. So they must be the same matrix. QED