Solutions Day 35, M 4/1/2024

Topic 16: Eigenstuff (day 2)

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Problem 1. *Suppose A is a 3×3 matrix with eigenvalue/vector pairs*

$$\begin{array}{ccccc} \lambda & = & 2 & 3 & 5 \\ \mathbf{v} & = & \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

(a) *Give the general solution to $\mathbf{x}' = A\mathbf{x}$, $\left(\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$.*

Solution: Since we already have the eigenpairs, the solution is immediate.

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{5t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) *Give the diagonalized form of A*

Solution: Again, with the eigenpairs, this is immediate

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad A = \underbrace{S\Lambda S^{-1}}_{\text{diagonal form}}.$$

(c) *Give A^5, A^{-1}, $\det(A)$.*

Solution: $A^5 = S\Lambda^5 S^{-1} = S \begin{bmatrix} 2^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & 5^5 \end{bmatrix} S^{-1}$.

$A^5 = S\Lambda^{-1} S^{-1} = S \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} S^{-1}$.

$\det A = \det \Lambda = 30$.

(d) *Give a change of variable that decouples the system in Part (a). Write the decoupled system in matrix form and solve it.*

Solution: Change of variable: $\mathbf{u} = S^{-1}\mathbf{x}$.

Decoupled system: $\mathbf{u}' = \Lambda \mathbf{u}$ or $\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ or $\begin{cases} u' & = 2u \\ v' & = 3v \\ w' & = 5w \end{cases}$

Eigenpairs for Λ (not really necessary to solve this system):

$$\begin{array}{ccccc} \lambda & = & 2, & 3, & 5 \\ \mathbf{v} & = & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array} \quad \text{(standard basis)}$$

So, $\mathbf{u} = c_1 e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3 e^{5t}\begin{bmatrix}0\\0\\1\end{bmatrix}$ or $\begin{cases} u &= c_1 e^{2t} \\ v &= c_2 e^{3t} \\ w &= c_3 e^{5t}\end{cases}$.

Problem 2. *Repeat problem (1) for the matrix* $\begin{bmatrix}5 & 3\\1 & 3\end{bmatrix}$.

Solution: We have to find the eigenvalue/vector pairs.

Characteristic equation: $|A - 2I| = \begin{vmatrix}5-\lambda & 3\\ 1 & 3-\lambda\end{vmatrix} = \lambda^2 - 8\lambda + 12 = 0.$

Eigenvalues (roots): $\lambda = 2,\ 6$.

Eigenspaces ($\text{Null}(A - \lambda I)$):

$\lambda = 2$: $A - 2I = \begin{bmatrix}3 & 3\\1 & 1\end{bmatrix} \xrightarrow{2\times 2 \text{ shortcut}}$ basic eigenvector $\begin{bmatrix}-3\\3\end{bmatrix}$ or $\begin{bmatrix}-1\\1\end{bmatrix}$.

$\lambda = 6$: $A - 6I = \begin{bmatrix}-1 & 3\\1 & -3\end{bmatrix} \xrightarrow{2\times 2 \text{ shortcut}}$ basic eigenvector $\begin{bmatrix}3\\1\end{bmatrix}$.

Summary of eigenpairs: $\boxed{\begin{array}{ccc} \lambda & = & 2, \quad 6 \\ \mathbf{v} & = & \begin{bmatrix}-1\\1\end{bmatrix}, \quad \begin{bmatrix}3\\1\end{bmatrix}\end{array}}$.

(a) General solution to $\mathbf{x}' = A\mathbf{x}$: $\mathbf{x}(t) = c_1 e^{2t}\begin{bmatrix}-1\\1\end{bmatrix} + c_2 e^{6t}\begin{bmatrix}3\\1\end{bmatrix}$.

(b) Diagonalization: $S = \begin{bmatrix}-1 & 3\\1 & 1\end{bmatrix}$, $\Lambda = \begin{bmatrix}2 & 0\\0 & 6\end{bmatrix}$, $A = S\Lambda S^{-1}$.

(c) $A^5 = S\Lambda^5 S^{-1} = S\begin{bmatrix}2^5 & 0\\0 & 6^5\end{bmatrix}S^{-1}$, $A^{-1} = S\Lambda^{-1}S^{-1} = S\begin{bmatrix}1/2 & 0\\0 & 1/6\end{bmatrix}S^{-1}$, $\det A = \det\Lambda = 12$.

(d) Change of variable: $\mathbf{u} = S^{-1}\mathbf{x}$.

Decoupled system: $\mathbf{u}' = \Lambda\mathbf{u}$ or $\begin{bmatrix}u'\\v'\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 6\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}$.

Solution to decoupled system: $\begin{bmatrix}u\\v\end{bmatrix} = c_1 e^{2t}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{6t}\begin{bmatrix}0\\1\end{bmatrix}$ or $\begin{cases} u &= c_1 e^{2t} \\ v &= c_2 e^{6t}\end{cases}$.

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