Day 36, T 4/2/2024

Topic 17: Matrix methods for DEs

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1 Agenda

- Systems with complex eigenvalues
- Companion systems
- Repeated roots
- Physical models: springs, populations, mixing tanks

2 Complex eigenvalues

Key: By linearity, both the real and imaginary parts of a complex solution are also solutions. (Just like ordinary DEs.)

Example 1. Let $A = \begin{bmatrix} -6 & 8 \\ -4 & 2 \end{bmatrix}$. Find the general real-valued solution to $\mathbf{x}' = A\mathbf{x}$.

Solution: Eigenstuff: $\left\{ \begin{array}{lll} \lambda: & -2+4i & -2-4i \\ \mathbf{v}: & \begin{bmatrix} 2 \\ 1+i \end{bmatrix} & \begin{bmatrix} 2 \\ 1-i \end{bmatrix} \end{array} \right\}$ (You should check this.)

One complex solution: $\mathbf{z}(t) = e^{(-2+4i)t} \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$

Expand this: $z(t) = e^{-2t}(\cos 4t + i \sin 4t) \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = e^{-2t} \begin{bmatrix} 2\cos 4t + i \cdot 2\sin 4t \\ (\cos 4t - \sin 4t) + i \cdot (\cos 4t + \sin 4t) \end{bmatrix}$

So, $\mathbf{x}_1(t) = \text{Re}(\mathbf{z}) = e^{-2t} \begin{bmatrix} 2\cos 4t \\ \cos 4t - \sin 4t \end{bmatrix}$ is a solution.

and $\mathbf{x}_2(t) = \text{Im}(\mathbf{z}) = e^{-2t} \begin{bmatrix} 2\sin 4t \\ \cos 4t + \sin 4t \end{bmatrix}$ is a solution.

General real-valued solution: $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$.

Example 2. Do problems 1, 2.

3 Companion systems

Example 3. Convert $x'' + 8x' + 7x = 0$ to a system of first-order equations.

Solution: Let $y = x'$. The DE becomes $y' + 8y + 7x = 0 \longrightarrow y' = -7x - 8y$.

The system is $\underbrace{\begin{array}{rcl} x' &=& y \\ y' &=& -7x-8y \end{array}}_{\text{2nd order companion system}}$ or $\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -7 & -8 \end{bmatrix}}_{\text{companion matrix}} \begin{bmatrix} x \\ y \end{bmatrix}$.

The second-order system is called the companion system to the original second-order DE. The coefficient matrix is called the companion matrix.

Example 4. Convert $2x''' + 3x'' + 4x' + 5x = 0$ to a system of first-order equations.

Solution: Let $y = x'$, $z = y' = x''$ $\Rightarrow$ $2z' + 3z + 4y + 5x = 0$. This gives

$$\underbrace{\begin{array}{rcl} x' &=& y \\ y' &=& z \\ z' &=& -5x/2 - 4y/2 - 3z/2 \end{array}}_{\text{companion system}} \quad \text{or} \quad \underbrace{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5/2 & -4/2 & -3/2 \end{bmatrix}}^{\text{companion matrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{companion system}}$$

Example 5. Do Problem 3.

4 Repeated roots: complete and defective systems

Example 6. Solve $\mathbf{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}$.

Solution: Diagonal matrix: Eigenpairs: $\left\{ \begin{array}{lcc} \lambda: & 2 & 2 \\ \mathbf{v}: & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array} \right.$.

So, $\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Second-order system, 2 independent eigenvectors $\longrightarrow$ system is complete

Example 7. Solve $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}$.

Solution: Eigenvalues: 2, 2

Eigenspace $= \text{Null}(A - 2I) = \text{Null}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$.

We find a basis of the eigenspace in the usual way:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\begin{array}{cc} \text{Free} & \text{Pivot} \\ 1 & 0 \end{array}$$

So the dimension $= 1$, basis $= \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We have one basic solution: $\mathbf{x}_1 = e^{2t}\begin{bmatrix}1\\0\end{bmatrix}$.

2×2 matrix with only one independent eigenvector $\longrightarrow$ system is defective.

Need another solution: Try $\mathbf{x} = te^{2t}\begin{bmatrix}1\\0\end{bmatrix} + e^{2t}\mathbf{w}$.

Plugging this into the DE and doing some algebra gives the requirement for $\mathbf{w}$:

$$(A - 2I)\mathbf{w} = \mathbf{v} \quad \longrightarrow \quad \text{can take } \mathbf{w} = \begin{bmatrix}0\\1\end{bmatrix}.$$

$\mathbf{w}$ is called a generalized eigenvector.

This is an edge case. Don't worry too much about the exact solution.

5 Population models

Example 8. Suppose country X has a natural growth rate of 4%/year and an emigration rate to Y of 2%/year. While Y has a natural growth rate of 3%/year and an emigration rate to X of 1%/year/

Model the populations x, y of X, Y and solve the resulting system.

Solution: Considering rates in and rates out, we have:

$$\begin{aligned} x' &= 0.04x - 0.02x + 0.01y \\ y' &= 0.03y - 0.01y + 0.02x \end{aligned} \longrightarrow \begin{aligned} x' &= 0.02x + 0.01y \\ y' &= 0.02x + 0.02y \end{aligned} \longrightarrow \begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}0.02 & 0.01\\0.02 & 0.02\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.$$

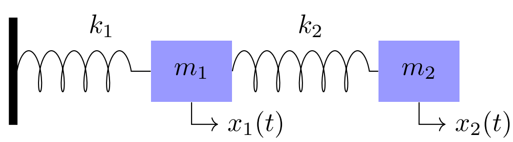
Rewrite the coefficient matrix as $\frac{1}{100}\begin{bmatrix}2 & 1\\2 & 2\end{bmatrix}$.

Find eigenpairs: $\begin{cases} \lambda : & \frac{1}{100}(2+\sqrt{2}) & \frac{1}{100}(2-\sqrt{2}) \\ \mathbf{v} : & \begin{bmatrix}1\\\sqrt{2}\end{bmatrix} & \begin{bmatrix}1\\-\sqrt{2}\end{bmatrix} \end{cases}$.

Solution: $\begin{bmatrix}x\\y\end{bmatrix} = \underbrace{c_1 e^{(2+\sqrt{2})t/100}\begin{bmatrix}1\\\sqrt{2}\end{bmatrix}}_{\text{dominant term}} + c_2 e^{(2-\sqrt{2})t/100}\begin{bmatrix}1\\-\sqrt{2}\end{bmatrix}$. Over time, the first term will dominate.

6 Coupled springs

Example 9. Consider the coupled springs shown. x_1, x_2 = displacement from equilibrium.



Our model is

$$m_1\ddot{x_1} = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2\ddot{x_2} = -k_2(x_2 - x_1)$$

This is two second-order equations or a fourth-order system.

Convert to a companion system of first-order equations.

Solution: Let $x_3 = \dot{x_1}$, $x_4 = \dot{x_2}$. The system becomes

$$\begin{array}{rcl} \dot{x_1} &=& x_3 \\ \dot{x_2} &=& x_4 \\ m_1\dot{x_3} &=& -(k_1+k_2)x_1 + k_2 x_2 \\ m_2\dot{x_4} &=& k_2 x_1 - k_2 x_2 \end{array} \longleftrightarrow \begin{array}{rcl} \dot{x_1} &=& x_3 \\ \dot{x_2} &=& x_4 \\ \dot{x_3} &=& -\dfrac{(k_1+k_2)}{m_1}\, x_1 + \dfrac{k_2}{m_1}\, x_2 \\ \dot{x_4} &=& \dfrac{k_2}{m_2}\, x_1 - \dfrac{k_2}{m_2}\, x_2 \end{array}$$

$$\longleftrightarrow \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2)/m_1 & k_2/m_2 & 0 & 0 \\ k_1/m_2 & -k_2/m_2 & 0 & 0 \end{bmatrix}}_{\text{companion matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

If time, do Problem 4.