Solutions Day 36, T 4/2/2024

Topic 17: Matrix methods for DEs

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Problem 1. *Let* $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2+i & 2-i \\ 3 & -4 & -4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3+4i & 0 \\ 0 & 0 & -3-4i \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2+i & 2-i \\ 3 & -4 & -4 \end{bmatrix}^{-1}$

Give the general real-valued solution to $\mathbf{x}' = A\mathbf{x}$.

Solution: A is given in diagonalized form. This immediately gives us eigenvalue/vector pairs.

$$\begin{aligned} \lambda &= -2, \quad -3+4i, \quad -3-4i \\ \mathbf{v} &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2+i \\ -4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2-i \\ -4 \end{bmatrix} \end{aligned}$$

So we have a real mode $\mathbf{x_1}(t) = e^{-2t} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

We also have the complex mode $\mathbf{z}(t) = e^{(-3+4i)t} \begin{bmatrix} 1 \\ 2+i \\ -4 \end{bmatrix}$.

We expand this to get two real solutions (real and imaginary parts)

$$\mathbf{z}(t) = e^{-3t}(\cos 4t + i \sin 4t) \begin{bmatrix} 1 \\ 2+i \\ -4 \end{bmatrix} = e^{-3t} \begin{bmatrix} \cos 4t & + & i \sin 4t \\ 2\cos 4t - \sin 4t & + & i(\cos 4t + 2\sin 4t) \\ -4\cos 4t & - & i4 \sin 4t \end{bmatrix}$$

So, $\mathbf{x_2}(t) = \text{Re}(\mathbf{z}) = e^{-3t} \begin{bmatrix} \cos 4t \\ 2\cos 4t - \sin 4t \\ -4\cos 4t \end{bmatrix}, \quad \mathbf{x_3}(t) = \text{Im}(\mathbf{z}) = e^{-3t} \begin{bmatrix} \sin 4t \\ \cos 4t + 2\sin 4t \\ -4\sin 4t \end{bmatrix}$.

The general solution is

$$\mathbf{x}(t) = c_1\mathbf{x_1} + c_2\mathbf{x_2} + c_3\mathbf{x_3} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} \cos 4t \\ 2\cos 4t - \sin 4t \\ -4\cos 4t \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} \sin 4t \\ \cos 4t + 2\sin 4t \\ -4\sin 4t \end{bmatrix}$$

Problem 2. *Solve* $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution: Eigenpairs $\boxed{\begin{aligned} \lambda &= 0, \quad 2 \\ \mathbf{v} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}}$ (Use our standard methods to find these.)

So, $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 3.
(a) *Give the companion system to $x''' + 2x'' + 4x' + 8x = 0$.*

Solution: Let $y = x'$, $z = x'' = y'$. The DE becomes $z' + 2z + 4y + 8x = 0$. The companion system is

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -8x - 4y - 2z \end{aligned} \qquad \Leftrightarrow \qquad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

(b) *What is the order of the companion system?*

Solution: Three first-order equations = 3rd order system.

Problem 4. *(Repeated eigenvalues: defective case – never on a quiz)*

Let $A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$. Solve $\mathbf{x}' = A\mathbf{x}$.

Solution: Eigenvalues: A is triangular, so $\lambda = 3, 3$.

Eigenvectors:

$$A - 3I = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{basic eigenvector} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\begin{matrix} \text{F} & \text{P} \\ 1 & 0 \end{matrix}$$

We have one modal solution: $\mathbf{x_1} = e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Generalized eigenvector: Solve $(A - 3I)\mathbf{w} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Take $\mathbf{w} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$. (Any other solution is okay.)

So, $\mathbf{x_2}(t) = te^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{3t} \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$.

General solution: $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$.