Day 39, F 4/5/2024

Topic 20: Step and delta functions (day 1)
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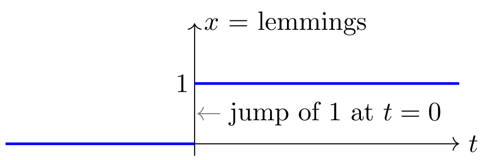
1 Agenda

- Definitely read the Topic 20 notes
- Back to $P(D)x = f(t)$ (ODE)
- Model impulses: finite change in an infinitesimal amount of time
- Unit step function: $u(t)$. Idealized and realistic versions
- Unit impulse: $\delta(t) = u'(t)$. Idealized and realistic versions.

2 Step functions

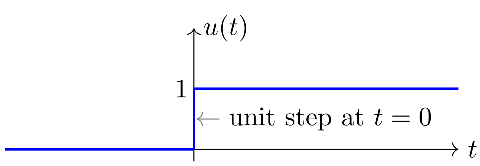
Example 1. Lemmings on the steppe. Let $x(t)$ be the size of the lemming population. Suppose that, prior to $t = 0$, we have $x = 0$. At $t = 0$, I add 1 unit of lemmings all at once. Here is the graph of $x(t)$.



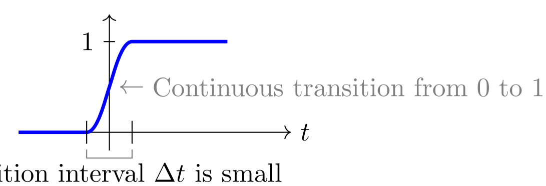
Rate of input: The rate lemmings are input at time 0 is $\dfrac{1 \text{ unit}}{0 \text{ time}} = \infty$. (Can think of the graph of $x(t)$ as being vertical at $t = 0$.) This type of input is called an impulse. The response is an instantaneous jump in some quantity.

Unit step function (Heaviside function):

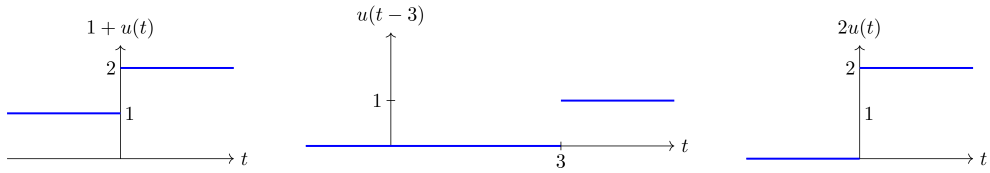
Idealized version:



Non-idealized version:



Variations:



3 0^+ and 0^-

When an impulse happens at $t = 0$, we need to distinguish between just before and just after the impulse:

$$\begin{array}{ll} \text{Just before 0 is } t = 0^-, & \text{e.g., } u(0^-) = 0 \\ \text{Just after 0 is } t = 0^+, & \text{e.g., } u(0^+) = 1 \end{array}$$

Technichally:

$$u(0^-) = \lim_{t \uparrow 0} u(t)$$
$$u(0^+) = \lim_{t \downarrow 0} u(t)$$

The jump at $t = 0$ is $u(0^+) - u(0^-) = 1$. Called a jump discontinuity.

4 Derivatives and integrals involving $u(t)$

$u'(t)$ doesn't exist at $t = 0$ in the 18.01 sense, but we can say "slope at $t = 0$ is infinity", i.e., "$u'(0) = \infty$".

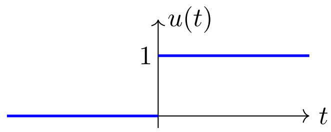
Ignoring technical issues with $u'(t)$, we know how integrals work.

$$\int u'(t)\,dt = u(t) + C$$
$$\int_{-5}^{5} u'(t)\,dt = u(t)\Big|_{-5}^{5} = u(5) - u(-5) = 1 - 0 = 1$$
$$\int_{-\infty}^{\infty} u'(t)\,dt = u(t)\Big|_{-\infty}^{\infty} = 1 \qquad \text{(Total area under } u'(t) \text{ is 1)}$$
$$\int_{5}^{10} u'(t)\,dt = u(10) - u(5) = 1 - 1 = 0$$
$$\int_{-\infty}^{-5} u'(t)\,dt = 0$$
$$\int_{0^-}^{10} u'(t)\,dt = u(10) - u(0^-) = 1 - 0 = 1$$
$$\int_{0^+}^{10} u'(t)\,dt = u(10) - u(0^+) = 1 - 1 = 0$$
$$\int_{0^-}^{0^+} u'(t)\,dt = u(0^+) - u(0^-) = 1 - 0 = 1$$

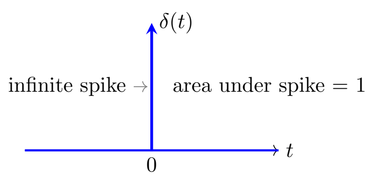
5 Unit impulse or delta function

Define $\delta(t) = u'(t)$ = unit impulse function = Dirac delta function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

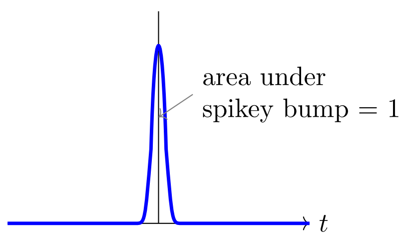


$$\delta(t) = \begin{cases} 0 & t < 0 \\ \infty & t = 0 \\ 0 & t > 0 \end{cases}$$



(Generalized function)

Non-idealized impulse:

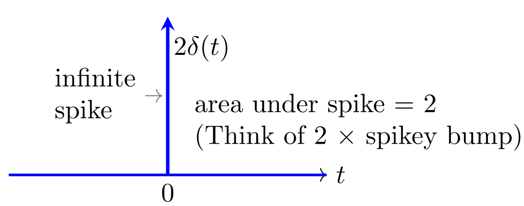


5.1 Scaling delta functions

Area under graph of $\delta(t) = \int_{-\infty}^{\infty} \delta(t)\, dt = 1$.

Area under graph of $2\delta(t) = \int_{-\infty}^{\infty} 2\delta(t)\, dt = 2$.

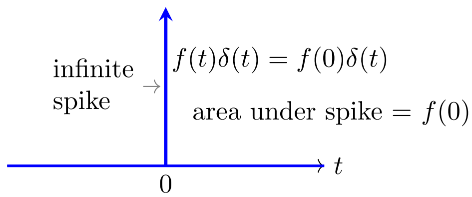
The graph of $2\delta(t)$ looks just like the graph of $\delta(t)$:



6 Integrals with delta functions (We like these!)

$$\int_a^b \delta(t)\, dt = u(b) - u(a) = \begin{cases} 0 & \text{if } 0 \text{ outside } [a, b] \\ 1 & \text{if } a < 0 < b \\ \text{undefined} & \text{if } a = 0 \text{ or } b = 0. \end{cases}$$

If $f(t)$ is continuous, then $\int_{-10}^{10} f(t)\, \delta(t)\, dt = f(0)$.



Idea: Since $\delta(t) = 0$ when $t \neq 0$, $f(t)\delta(t) = f(0)\delta(t)$.

That is, imagine the non-idealized δ scaled by $f(t)$. Non-ideal δ is only non-zero in a small interval around 0. In this interval, $f(t) \approx f(0)$. So, $f(t)\cdot$non-ideal $\delta(t) \approx f(0)\cdot$non-ideal $\delta(t)$. As the interval shrinks to 0, we get our formula $f(t)\delta(t) = f(0)\delta(t)$.

Likewise, $\displaystyle\int_{0^-}^{0^+} f(t)\delta(t)\,dt = f(0)$, $\displaystyle\int_5^{10} f(t)\delta(t)\,dt = 0$, $\displaystyle\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$.

7 Delta functions as input to a DE

(Reasons in Topic 20 notes. We'll discuss this tomorrow.)

Example 2. Solve $x' + 10x = \delta(t)$, $x(0^-) = 3$.

$x(0^-) = 3$ is called a pre-initial condition, i.e., before the impulse at time 0.

Solution: Solve in cases: $t < 0$, $t > 0$.

Case $t < 0$ (before impulse):

DE: $x' + 10x = 0$ (input is 0 since $\delta(t) = 0$ for $t < 0$)

Pre-IC: $x(0^-) = 3$

Solving: $x(t) = Ce^{-10t}$. $x(0^-) = C = 3$. So, for $t < 0$, $x(t) = 3e^{-10t}$.

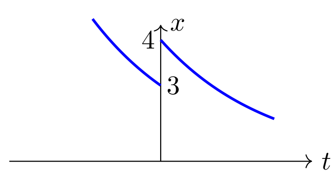
Case $t > 0$ (after impulse):

DE: $x' + 10x = 0$ (input is 0 since $\delta(t) = 0$ for $t < 0$)

Post-IC: $x(0^+) = 3 + 1 = 4$ (+1: effect of impulse at $t = 0$ –discuss tomorrow)

Solving: $x(t) = Ce^{-10t}$. $x(0^+) = C = 4$. So, for $t > 0$, $x(t) = 4e^{-10t}$.

All together: $x(t) = \begin{cases} 3e^{-10t} & \text{for } t < 0 \\ 4e^{-10t} & \text{for } t > 0. \end{cases}$



Example 3. (2nd order DE)

Solve $\underbrace{4}_{\text{mass}} x'' + 4x' + 5x = \underbrace{3\delta(t)}_{\text{Input = force = impulse of magnitude 3}}$, $\underbrace{x(0^-) = 1,\ x'(0^-) = 2}_{\text{Pre-initial conditions}}$.

Solution: Force changes momentum over time. So an impulse causes a jump in momentum, i.e., mx' jumps by 3 at $t = 0$.

Solve in cases.

Case $t < 0$ (before impulse):

DE: $4x'' + 4x' + 5x = 0$ (input is 0 since $\delta(t) = 0$ for $t < 0$)

Pre-IC: $x(0^-) = 1,\ x'(0^-) = 2$

Do the usual arithmetic to show: For $t < 0$: $x(t) = e^{-t/2}\cos t + \dfrac{5}{2}e^{-t/2}\sin t$. (Won't show arithmetic, because that is not the point of this example.)

Case $t > 0$ (after impulse):

DE: $4x'' + 4x' + 5x = 0$ (input is 0 since $\delta(t) = 0$ for $t < 0$)

Post-IC: $\begin{cases} x(0^+) = x(0^-) = 1 & \text{(Impulse doesn't affect position in 2nd order system.)} \\ x'(0^+) = x(0^-) + 3/4 = 2.75 & \text{(Impulse causes a jump of 3 in momentum,} \\ & \text{so causes a jump of 3/mass in velocity.)} \end{cases}$

Usual arithmetic: For $t > 0$: $x(t) = e^{-t/2}\cos t + 3.25e^{-t/2}\sin t$.

All together: $x(t) = \begin{cases} e^{-t/2}\cos t + 2.5e^{-t/2}\sin t & \text{for } t < 0 \\ e^{-t/2}\cos t + 3.25e^{-t/2}\sin t & \text{for } t > 0. \end{cases}$