Solutions Day 39, F 4/5/2024
Topic 20: Step and delta functions
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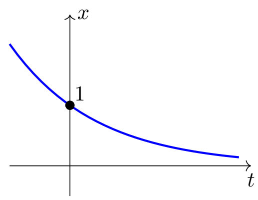
Problem 1. *Solve $x' + 5x = 0$, $x(0) = 1$. Graph the solution.*

Solution: General solution: $x(t) = \mathbf{C}e^{-5t}$. Using the IC: $x(0) = C = 1$. So, $x(t) = e^{-5t}$.



$x(t) = e^{-5t}$

Problem 2. *Solve $x' + 5x = 2\delta(t)$, $x(0^-) = 0$. Graph the solution.*

Solution: Work in cases.

<u>Case $t < 0$</u> (pre-impulse)

DE: $x' + 5x = 0$, Pre-IC: $x(0^-) = 0$.

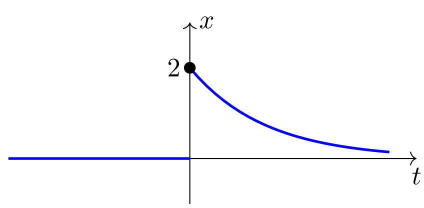
So, $x(t) = 0$ (easy to check).

<u>Case $t > 0$</u> (post-impulse)

DE: $x' + 5x = 0$, Post-IC: $x(0^+) = \overbrace{x(0^-) + 2}^{2 \,=\, \text{size of impulse}} = 2$.

So, $x(t) = Ce^{-5t}$. $x(0^+) = C = 2 \quad \Rightarrow x(t) = 2e^{-5t}$.

Answer: $x(t) = \begin{cases} 0 & t < 0 \\ 2e^{-5t} & t > 0 \end{cases}$.



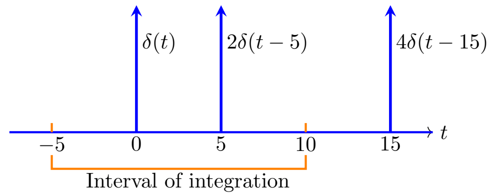
Problem 3. *Compute each of the following integrals.*

(a) $\displaystyle\int_{-\infty}^{\infty} \delta(t)\,dt.$

Solution: Area under $\delta(t) = \boxed{1}$.

(b) $\displaystyle\int_{-5}^{10} \delta(t) + 2\delta(t-5) + 4\delta(t-15)\,dt.$

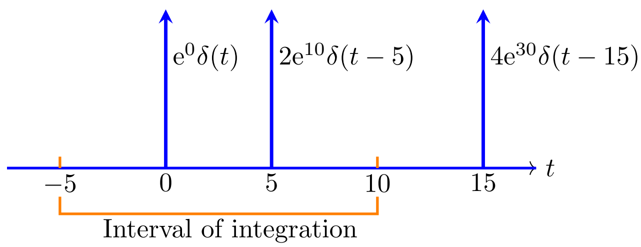
Solution: The integral includes the area under the spikes at $t = 0$ and $t = 5$, but not the spike at $t = 15$.



So the integral $= 1 + 2 = \boxed{3}$.

(c) $\int_{-5}^{10} e^{2t}\left[\delta(t) + 2\delta(t-5) + 4\delta(t-15)\right]\, dt.$

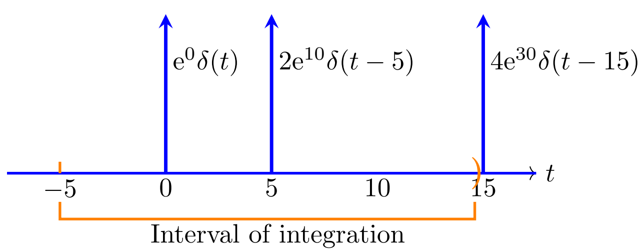
Solution: In general, $\int_{-5}^{10} f(t)\left[\delta(t) + 2\delta(t-5) + 4\delta(t-15)\right]\, dt = f(0) + 2f(5).$



For this problem, $f(t) = e^{2t}$. So, integral $= \boxed{1 + 2e^{10}}$.

(d) $\int_{-5}^{15^-} e^{2t}\left[\delta(t) + 2\delta(t-5) + 4\delta(t-15)\right]\, dt.$

Solution: For this problem, the interval of integration has changed from Part (c). But it still doesn't contain the spike at $t = 15$.



So we get the same answer as in Part (c): $\boxed{1 + 2e^{10}}$.

(e) $\int_{-\infty}^{\infty} e^{\tan t}\delta(t - \pi/4)\, dt$

Solution: $\int_{-\infty}^{\infty} e^{\tan t}\delta(t - \pi/4)\, dt = e^{\tan(\pi/4)} = \boxed{e^1 = e}$.

Problem 4.

(a) *Find the general solution to $2x'' + 7x' + 6x = 0$. (Roots $= -3/2, -2$.)*

Solution: The roots are given, so $x(t) = c_1 e^{-3t/2} + c_2 e^{-2t}$.

(b) *Solve $2x'' + 7x' + 6x = 6\delta(t), \quad x(0^-) = 0, \quad x'(0^-) = 0.$*

Solution: Solve in cases.

Case $t < 0$ (pre-impulse)

$$\left.\begin{array}{ll}\text{DE:} & 2x'' + 7x' + 6x = 0 \\ \text{Pre-IC:} & x(0^-) = 0,\ x'(0^-) = 0\end{array}\right\} \quad x(t) = 0 \text{ for } t < 0 \quad \text{(easy to check).}$$

Case $t > 0$ (post-impulse)

$$\begin{array}{rl}\text{DE:} & 2x'' + 7x' + 6x = 0 \\ \text{Post-IC:} & x(0^+) = x(0^-) = 0, \quad x'(0^+) = \underbrace{x'(0^-) + 6/2}_{6/2\,=\,\text{size of impulse/leading coeff.}} = 3\end{array}$$

From Part (a), $x(t) = c_1 e^{-3t/2} + c_2 e^{-2t}$. We use the post-IC to find c_1, c_2.

$$\left.\begin{array}{ll} x(0^+) & = c_1 + c_2 = 0 \\ x'(0^+) & = -\frac{3}{2}\, c_1 - 2c_2 = 3\end{array}\right\} \quad \text{solving gives } c_1 = 6,\ c_2 = -6.$$

So, $\boxed{x(t) = \begin{cases} 0 & t < 0 \\ 6e^{-3t/2} - 6e^{-2t} & t > 0\end{cases}}$