Solutions Day 4, R 2/8/2024

Topic 2: Linear Systems, input response

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Problem 1. *Say whether or not the following equations are first-order linear DEs.*

(a) $t^2y' + 2ty = 3y^2$

(b) $y' + 7y = \tan t$

(c) $(y' + y)t = 3$

(d) $y' + e^y y = 3t$

Solution: (a) Not linear: The $3y^2$ term is not linear.

(b) Linear: this is in standard form $y' + p(t)y = q(t)$.

(c) Linear: $ty' + ty = 3$ is in general form.

(d) Nonlinear: The $e^y y$ term is not linear.

Problem 2. *Solve each of the following.*

(a) $y' + 5y = 0$ *(homogeneous).*

Solution: $y' = -5y$: This is an exponential decay equation. So, $y(t) = Ce^{-5t}$.

(b) $y' + 5y = 5, \quad y(0) = 0.$

Solution: This is first-order linear in standard form.

Associated homogeneous: $y_h + 5y_h = 0$. From Part (a), $y_h(t) = e^{5t}$. (Don't need C.)

Variation of parameters formula:

$$\begin{aligned} y(t) &= y_h(t)\left[\int \frac{q(t)}{y_h(t)}\,dt + C\right] = y_h(t)\int \frac{q(t)}{y_h(t)}\,dt + Cy_h(t) \\ &= e^{-5t}\int \frac{5}{e^{-5t}}\,dt + Ce^{-5t} \\ &= e^{-5t}\int 5e^{5t}\,dt + Ce^{-5t} \\ &- e^{-5t}\cdot e^{5t} + Ce^{-5t} \\ &= 1 + Ce^{-5t}. \end{aligned}$$

Using the initial condition (IC): $y(0) = 1 + C \quad \Rightarrow C = -1$.

So, $\boxed{y(t) = 1 - e^{-5t}}$.

(c) $y' + 5y = t$.

Solution: This is similar to Part (b): $y(t) = e^{-5t}\int te^{5t}\,dt + Ce^{-5t}$.

Using integration by parts:

$$y(t) = e^{-5t}\left(\frac{te^{5t}}{5} - \frac{e^{5t}}{25}\right) + Ce^{-5t} = \boxed{\frac{t}{5} - \frac{1}{25} + Ce^{-5t}}.$$

(d) $y' + 5y = 17 + 9t$.

Solution: The input $17+9t = \frac{17}{5}\cdot 5+9\cdot t$ is a linear combination of the inputs from (b) and (c). So the solution is the same linear combination of their solutions: $\boxed{y(t) = \frac{17}{5} + 9\left(\frac{t}{5} - \frac{1}{25}\right) + Ce^{-5t}}$.

Problem 3.

(a) *State the superposition principle for a first-order linear DE.*

Solution: Superposition principle: A superposition (linear combination) of the inputs to a linear DE has solution using the same linear combination of the solutions.

More precisely: Suppose $\left.\begin{aligned} y_1' + p(t)y_1 &= q_1(t) \\ y_2' + p(t)y_2 &= q_2(t) \end{aligned}\right\}$ same $p(t)$, then $y = c_1y_1 + c_2y_2$ is a solution to $y' + p(t)y = c_1q_1(t) + c_2q_2(t)$.

Here, c_1, c_2 are constants.

(b) *Show that the nonlinear DE $y'y = q(t)$ does not satisfy the superposition principle. ($q(t)$ = input, can be any function.)*

Solution: We have to show that a statement analagous to the one in Part (a) is **not** true.

Suppose $y_1'y_1 = q_1$, i.e., y_1 solves $y'y = q_1$. Likewise, suppose $y_2'y_2 = q_2$.

Now, we have to see that $y = c_1y_1 + c_2y_2$ does not solve $y'y = c_1q_1 + c_2q_2$. To do this, we just plug y into the DE

$$\begin{aligned} y'y &= (c_1y_1 + c_2y_2)'(c_1y_1 + c_2y_2) \\ &= c_1^2\underbrace{y_1'y_1}_{q_1} + c_1c_2y_1'y_2 + c_1c_2y_1y_2' + c_2^2\underbrace{y_2'y_2}_{q_2} \\ &= c_1^2q_1 + c_1c_2(y_1'y_2 + y_1y_2') + c_2^2q_2 \end{aligned}$$

This clearly does not equal $c_1q_1 + c_2q_2$.

To show this easily, let $c_1 = 2$, $c_2 = 0$. Then, we get $y'y = c_1^2q_1 = 4q_1 \neq 2q_1$, i.e., $y'y \neq c_1q_1$.

(c) *Show that $y' + p(t)y = q(t)$ does satisfy the superposition principle.*

Solution: For this we just plug and chug with the DE.

Suppose $y_1' + py_1 = q_1$ and $y_2' + py_2 = q_2$.

Let $y = c_1y_1 + c_2y_2$. We plug this into $y' + py$:

$$\begin{aligned} y' + py &= (c_1y_1 + c_2y_2)' + p(c_1y_1 + c_2y_2) \\ &= c_1y_1' + c_2y_2' + c_1py_1 + c_2py_2 \\ &= c_1\underbrace{y_1' + py_1}_{q_1} + c_2\underbrace{y_2' + py_2}_{q_2} \\ &= c_1q_1 + c_2q_2 \quad \text{(follows from assumption } y_1' + py_1 = q_1,\ y_2' + py_2 = q_2) \end{aligned}$$

Thus, y solves $y' + py = c_1q_1 + c_2q_2$. ■

Problem 4. *Solve* $y' + 5y = \begin{cases} 0 & \textit{for } t < 0 \\ 5 & \textit{for } 0 < t < 1 \\ t & \textit{for } 1 < t \end{cases}$ *, with* $y(0) = 0$*,* $y(t)$ *is continuous.*

Solution: The input is given in cases. Each case was solved in Problem 2. The new wrinkle is that we have to patch together the cases so that y is continuous. We work one case at a time.

Case (i) $t < 0$: $y' + 5y = 0, \quad y(0) = 0$.

Solution $y(t) = Ce^{-5t}$.

Initial condition: $y(0) = 0 = C$. So, $y(t) \equiv 0$ for $t < 0$.

Case (ii) $0 < t < 1$: $y' + 5y = 5, \quad y(0) = 0$.

From 2(b): $y(t) = 1 - e^{-5t} \quad$ for $0 < t < 1$.

In preparation for the next case: $y(1) = 1 - e^{-5}$.

Case (iii) $1 < t$: $y' + 5y = t, \quad y(1) = 1 - e^{-5}$.

From 2(c): $y(t) = \dfrac{t}{5} - \dfrac{1}{25} + Ce^{-5t}$.

Using the initial condition: $y(1) = \dfrac{1}{5} - \dfrac{1}{25} + Ce^{-5} = 1 - e^{-5} \quad \Rightarrow C = \dfrac{21}{25}e^5 - 1$.

Putting this together, $\quad y(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 - e^{-5t} & \text{for } 0 < t < 1 \\ \frac{t}{5} - \frac{1}{25} + \left(\frac{21}{25}e^5 - 1\right)e^{-5t} & \text{for } 1 < t \end{cases}$

Problem 5. *Consider the family of ellipses* $x^2 + 2y^2 = c$*. Find an orthogonal family of curves.*

Solution: Differentiation of the equation for the ellipse with respect to x gives

$$2x + 4y\frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{x}{2y}.$$

For the orthogonal family, $\dfrac{dy}{dx} = \dfrac{2y}{x}$ (negative reciprocal).

This is separable $\quad \Rightarrow \dfrac{dy}{y} = \dfrac{2\,dx}{x}$.

Integrate: $\ln|y| = 2\ln|x| + C = \ln(x^2) + C$.

Algebra: $|y| = e^C x^2$. $\quad$ So, $y = \pm e^C x^2$.

Lost solution: $y(x) \equiv 0$.

Altogether, all solutions are of the form $y = Cx^2$, where C takes any value. That is, we have a family of parabolas.