Day 41, T 4/9/2024

Topic 20: Delta functions (day 2)
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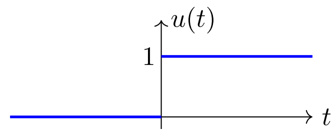
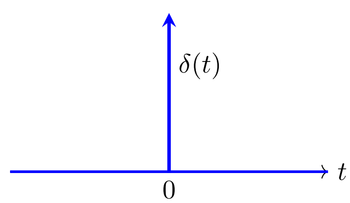
1 Agenda

- Impulses: add something all at once, e.g., lemmings or momentum
- $\delta(t)$ as impulsive input to DEs
- Generalized derivatives
- Algebraic explanation for post-initial conditions after impulse. (If time.)

Unit step function $u(t)$
(See notes from last class.)



Unit impulse function $\delta(t) = u'(t)$
(See notes from last class.)



2 Solving $P(D)x =$ impulse: mechanics of solving

Do problems 1, 2, 4 from last time.

3 Physical impulses

Impulse: input all at once.

- Ideally: input rate $= \infty$ over infinitesimal time. Total amount input is finite.
- Non-ideally: input rate is very large over a very short time.
- Causes a jump in whatever is being input.
- For a unit jump, the rate of input is $\dfrac{du}{dt} = \delta(t)$. This has dimension $\dfrac{\text{something}}{\text{time}}$.

3.1 First-order system

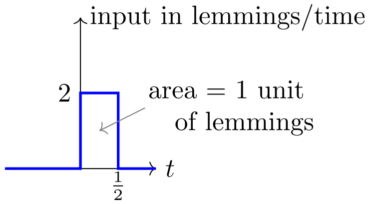
First-order system: $x' + kx = \delta(t)$. Say $x =$ lemmings, then $\delta(t)$ is in $\frac{\text{lemmings}}{\text{time}}$.

Input at rate $\delta(t)$ causes a unit jump in the number of lemmings between between $t = 0^-$ and $t = 0^+$.

Likewise, input $\delta(t-a)$ would cause a unit jump between times $t = a^-$ and $t = a^+$.

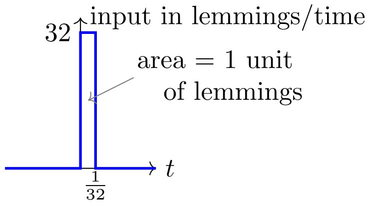
3.2 Unit impulse as a limit of boxes

Say $f(t) =$



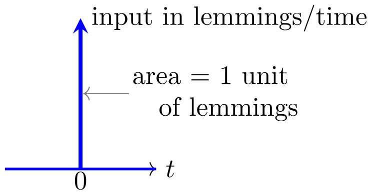
Total input from $t = 0$ to $1/2$ is $\displaystyle\int_0^{1/2} f(t)\,dt = 1$ unit of lemmings.

Say $f(t) =$



Total input from $t = 0$ to $1/32$ is $\displaystyle\int_0^{1/32} f(t)\,dt = 1$ unit of lemmings.

Say $f(t) = \delta(t) =$



Total input from $t = 0^-$ to 0^+ is $\displaystyle\int_{0^-}^{0^+} \delta(t)\,dt = 1$ unit of lemmings.

3.3 Second-order systems (springs)

For a damped spring system $mx'' + bx' + kx = f(t)$, the input has units of force. As a rate,

$$f(t) = \text{ force } = \frac{\text{kg}\cdot\text{m/sec}}{\text{sec}} = \frac{\text{momentum}}{\text{sec}}.$$

That is, force adds momentum over time, i.e., it changes mx'.

Example 1. Consider $mx'' + bx' + kx = 5\delta(t)$, $x(0^-) = b_0$, $x'(0^-) = b_1$.

Give the pre impulse and post impulse initial conditions.

Solution: Pre-IC: $x(0^-) = b_0$, $x'(0^-) = b_1$ (These were given to us.)

The impulse causes no change in position. So, $x(0^+) = x(0^-) = b_0$.

The impulse adds 5 units of momentum. So,

$$\underbrace{mx'(0^+)}_{\substack{\text{Post-IC}\\\text{momentum}}} = \underbrace{mx'(0^-)}_{\substack{\text{Pre-IC}\\\text{momentum}}} + \underbrace{5}_{\text{impulse}} \longrightarrow \underbrace{x'(0^+)}_{\substack{\text{Post-IC}\\\text{velocity}}} = x'(0^-) + 5/m = b_1 + 5/m.$$

Input causes a jump in momentum not position.

3.4 Third-order systems

Third-order system: $ax''' + bx'' + cx' + dx = 5\delta(t)$.

Note: $5\delta(t)$ has the same units as ax''', i.e., units of ax''/time. So input adds units of ax'' over time.

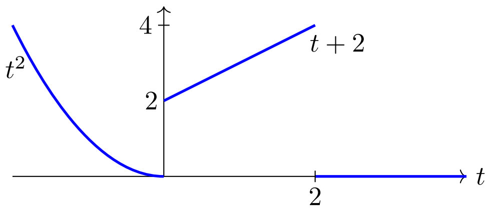
Our post-IC are

$$\begin{array}{lcll} x(0^+) & = & x(0^-) & \text{(no jump in } x\text{)} \\ x'(0^+) & = & x'(0^-) & \text{(no jump in } x'\text{)} \\ x''(0^+) & = & x''(0^-) + 5/a & \text{(jump of 5 in } ax''\text{)} \end{array}$$

4 Generalized derivatives

A jump discontinuity causes the derivative to have an impulse. The magnitude of the impulse is the size of the jump.

Example 2. Compute the generalized derivative of



Solution:
$$\underbrace{f'(t)}_{\text{Generalized deriv.}} = \underbrace{\begin{cases} 2t & t < 0 \\ 1 & 0 < t < 2 \\ 0 & 2 < t \end{cases}}_{\text{Regular part}} + \underbrace{2\delta(t) - 4\delta(t-2)}_{\substack{\text{Singular part} \\ \text{(due to jumps)}}}$$

5 Algebraic explanation of post-initial conditions

See Topic 20 notes. If there is time we will discuss this in class.