Solutions Day 41, T 4/9/2024

Topic 20: Delta functions (day 2)

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Problem 1. *Compute $g'(t)$. Identify the regular and singular parts.*

(a) $g(t) = \begin{cases} t & \text{for } t < 0 \\ t^2+1 & \text{for } 0 < t < 2 \\ 5 & \text{for } 2 < t < 4 \\ t^2 - 4t & \text{for } 4 < t \end{cases}.$

Solution:

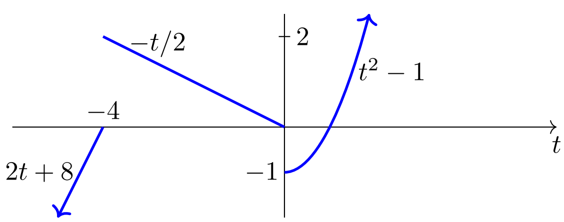
$$\begin{aligned} g(0^-) &= 0, \quad g(0^+) = 1 \longrightarrow \text{ jump of 1} \\ g(2^-) &= 5, \quad g(2^+) = 5 \longrightarrow \text{ no jump} \\ g(4^-) &= 5, \quad g(4^+) = 0 \longrightarrow \text{ jump of -5} \end{aligned}$$

So,

$$g'(t) = \underbrace{\begin{cases} 1 & \text{for } t < 0 \\ 2t & \text{for } 0 < t < 2 \\ 0 & \text{for } 2 < t < 4 \\ 2t-4 & \text{for } 4 < t \end{cases}}_{\text{regular part}} + \underbrace{\delta(t) - 5\delta(t-4)}_{\text{singular part}}$$

(b)

$g(t) =$



Solution: The graph shows

$$\begin{aligned} g(-4^-) &= 0, \quad g(4^+) = 2 \longrightarrow \text{ jump of 2} \\ g(0^-) &= 0, \quad g(0^+) = -1 \longrightarrow \text{ jump of -1} \end{aligned}$$

So,

$$g'(t) = \underbrace{\begin{cases} 2 & \text{for } t < -4 \\ -1/2 & \text{for } -4 < t < 0 \\ 2t & \text{for } 0 < t \end{cases}}_{\text{regular part}} + \underbrace{2\delta(t+4) - \delta(t)}_{\text{singular part}}$$

Problem 2. *Solve* $3x''' + 18x'' + 33x' + 18x = \delta(t-4)$, $\overbrace{x(0^-) = 0,\ x'(0^-) = 0,\ x''(0^-) = 0}^{\text{rest initial conditions}}$

Hint: characteristic roots are -1, -2, -3.

Get to the point of setting up equations for the coefficients c_1, c_2, c_3, but don't solve for them.

Solution: The impulse is at $t = 4$, so that determines our cases.

Case $t < 4$	DE:	$3''' + 18x'' + 33x' + 18x = 0$
	Pre-IC:	$x(0^-) = 0,\ x'(0^-) = 0,\ x''(0^-) = 0$
	Solution:	$x(t) = 0$ for $t < 4$
	Thus,	$x(4^-) = 0,\ x'(4^-) = 0,\ x''(4^-) = 0.$

Case $t > 4$	DE:	$3''' + 18x'' + 33x' + 18x = 0$
	Pre-IC:	$x(4^-) = 0$
		$x'(4^-) = 0$
		$x''(4^-) = 0$
	Post-IC:	$x(4^+) = x(4^-) = 0$
		$x'(4^+) = x'(4^-) = 0$
		$3x''(4^+) = 3x''(4^-) + 1 = 1 \quad \Rightarrow \quad x''(4^+) = 1/3$

Given the roots, we know $x(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}$.

Using the post-IC we have

$$\begin{aligned} x(4^+) &= c_1 e^{-4} + c_2 e^{-8} + c_3 e^{-12} \\ x'(4^+) &= -c_1 e^{-4} - 2c_2 e^{-8} - 3c_3 e^{-12} \\ x''(4^+) &= c_1 e^{-4} + 4c_2 e^{-8} + 9c_3 e^{-12} \end{aligned}$$

We can solve for c_1, c_2, c_3. If we're careful, we'll find

$$x(t) = \frac{1}{6}e^{-(t-4)} - \frac{1}{3}e^{-2(t-4)} + \frac{1}{6}e^{-3(t-4)} \qquad \text{for } t > 4.$$

Thus, $x(t) = \begin{cases} 0 & \text{for } t < 4 \\ \frac{1}{6}e^{-(t-4)} - \frac{1}{3}e^{-2(t-4)} + \frac{1}{6}e^{-3(t-4)} & \text{for } t > 4. \end{cases}$