Day 43, R 4/11/2024

**Topic 21: Fourier series (day 1)**

Jeremy Orloff

# 1 Agenda

- Reminder: superposition principle
- Periodic functions
- Fourier Theorem/Series
- Square wave

# 2 Summation notation

We are going to make heavy use of summation notation $\sum_{i=1}^{n}$. You need to be completely fluent with this.

# 3 Superposition principle

Suppose $f(t)$ is periodic. Then:

Fourier: $f(t)$ is a linear combination of sines and cosines (usually an infinite number of them).

DEs: $P(D)x = f(t)$ can then be solved by the SRF and superposition principle.

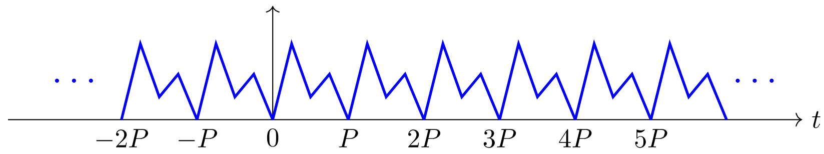
**Example 1.** Solve $x'' + 8x' + 7x = \cos t + \dfrac{\cos 2t}{4} + \dfrac{\cos 3t}{9} + \dfrac{\cos 4t}{16} + \ldots$

**Solution:** We solve one piece at a time and then use superposition.

$$\left.\begin{aligned} x_1'' + 8x_1' + 7x_1 &= \cos t \\ x_2'' + 8x_2' + 7x_2 &= \cos 2t/4 \\ x_3'' + 8x_3' + 7x_3 &= \cos 3t/9 \\ &\ldots \end{aligned}\right\} x(t) = x_1(t) + x_2(t) + x_3(t) + \ldots \quad \text{(superposition principle)}$$

# 4 Periodic functions

$f(t)$ is periodic with period $p > 0$ means $f(t+p) = f(t)$ for all $t$, i.e., $f(t)$ repeats every $p$ units. Graphically:




## 4.1 Basic periodic functions

$\cos t$, $\sin t$: period: $\underbrace{2\pi}_{\text{Base period}}$, $4\pi$, $6\pi$, ... Angular frequency = 1 (in radians/sec).

$\cos(\omega t)$, $\sin(\omega t)$: period: $\underbrace{2\pi/\omega}_{\text{Base period}}$, $4\pi/\omega$, $6\pi/\omega$, ... Angular frequency = $\omega$.

1: period: any $p > 0$.

Note: Engineers use frequency $f$ in cycles/time. Since there are $2\pi$ radians per cycle: $\omega = 2\pi f$ or $f = \omega/2\pi$.

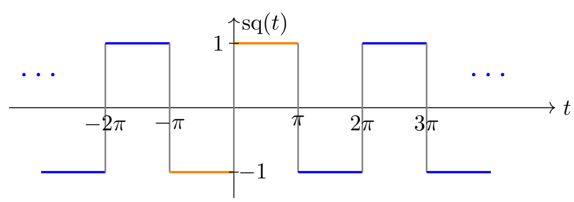
## 4.2 Specifying a periodic function

Step 1. Give the period

Step 2. Define the function over 1 period.

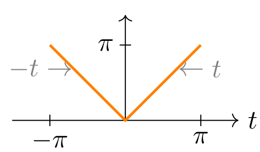
**Example 2.** (Standard, period $2\pi$ square wave, sq($t$))

Period = $2\pi$. Over one period $\text{sq}(t) = \left\{ \begin{array}{ll} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{array} \right\}$ $-\pi$ to $\pi$ = one period.

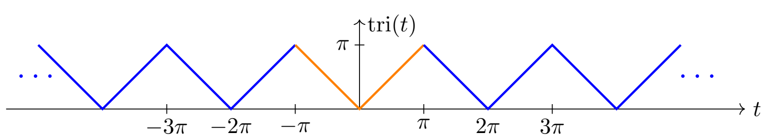



**Example 3.** (Standard, period $2\pi$ triangle wave, tri($t$))

Period $2\pi$. Over one period tri($t$) =

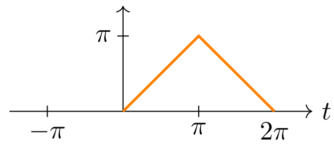


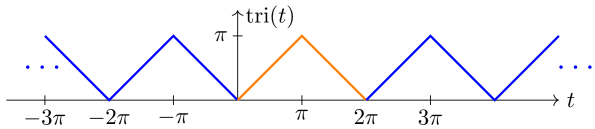

So tri($t$) has graph:




Note: over $[-\pi, \pi]$, tri($t$) = $|t|$.

We could also define tri($t$) over the period $[0, 2\pi]$:




# 5 Simple handy formulas

$\cos(n\pi) = (-1)^n, \quad n = 0, 1, 2, 3, ...$

$\sin(n\pi) = 0, \quad n = 0, 1, 2, 3, ...$

# 6 Fourier Theorem

**Fourier Theorem:** If $f(t)$ has period $2L$ (So, $L = 1/2$ period), then

$$\begin{aligned} f(t) &= \frac{a_0}{2} + a_1 \cos\left(\frac{\pi}{L}\, t\right) + a_2 \cos\left(\frac{2\pi}{L}\, t\right) + a_3 \cos\left(\frac{3\pi}{L}\, t\right) + ... \\ &\qquad + b_1 \sin\left(\frac{\pi}{L}\, t\right) + b_2 \sin\left(\frac{2\pi}{L}\, t\right) + b_3 \sin\left(\frac{3\pi}{L}\, t\right) + ... \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\, t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\, t\right) \end{aligned}$$

(This is called the Fourier series for $f(t)$.)

where,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi}{L}\, t\right)\, dt, \qquad a_0 = \frac{1}{L} \int_{-L}^{L} f(t)\, dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi}{L}\, t\right)\, dt$$

(These are the formulas for the Fourier coefficients.)

Notes:

- $a_n =$ coefficient of $\cos\left(\frac{n\pi}{L}\, t\right)$.
- $b_n =$ coefficient of $\sin\left(\frac{n\pi}{L}\, t\right)$.
- The integral is over one period, $[-L, L]$. Any other period will do, e.g., $[0, 2L]$ or even $[3, 3 + 2L]$.

### 6.1 Terminology

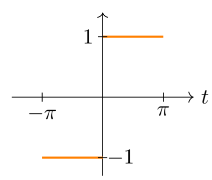
1. $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ = Fourier coefficients of $f(t)$.
2. $L = 1/2$ period. $2L$ = period.
3. All the functions $\cos\left(\frac{n\pi}{L}t\right)$, $\sin\left(\frac{n\pi}{L}t\right)$ have period $2L$.
4. $2L$ is called the base period
5. $\frac{\pi}{L}$ = fundamental frequency = first harmonic frequency
   $\frac{n\pi}{L}$ = $n$th harmonic frequency (All frequencies are multiples of the fundamental freq.)
6. $\frac{a_0}{2}$ = DC term = direct current term.

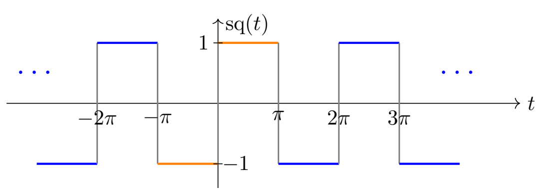
**Notes:**

1. $\frac{a_0}{2}$ is a convention. It makes the integral formuls uniform. Forgetting the 1/2 is a common source of errors.
2. The proof of the Fourier Theorem requires the following two things:
   Formulas for the coefficients: See Topic 22 notes. (Uses the "orthogonality relations").
   Showing every periodic function has a Fourier series is more difficult. It is in an "enrichment note" posted with the topic notes.
3. $L = \pi$ (period $2\pi$) is nice because $\frac{n\pi}{L} = n$.

### 6.2 Standard square wave

sq($t$): period $2\pi$, over one period: $\text{sq}(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$ =




In the problems we'll see that

$$\text{sq}(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \ldots\right) \quad \text{(The ... is important.)}$$
$$= \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt)}{n}.$$

### 6.3 Fourier approximation applet

https://web.mit.edu/jorloff/www/OCW-ES1803/fourierapproximation.html