Solutions Day 43, R 4/11/2024

Topic 21: Fourier Series (day 1)

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Problem 1. *Show* $\cos\left(\frac{n\pi}{3}t\right)$*,* $n = 1, 2, 3, \ldots$ *all have a common period.*

Solution: $\cos(\omega t)$ has period $2\pi/\omega$, so $\cos\left(\frac{n\pi}{3}t\right)$ has period $\dfrac{2\pi}{n\pi/3} = \dfrac{6}{n}$.

Since any multiple of a period is also a period, it also has period $n \cdot \frac{6}{n} = 6$. That is, all the functions $\cos\left(\frac{n\pi}{3}t\right)$ have 6 as a common period.

Problem 2.

(a) *Write out the sequence* $\cos(n\pi)$*,* $n = 0, 1, 2, \ldots$*. Write it out in a simple way in terms of* n*.*

Solution: We have the following table

n	0	1	2	3	4	5	...
$\cos(n\pi)$	1	−1	1	−1	1	−1	...

$\Rightarrow \cos(n\pi) = (-1)^n$.

(b) *Same question for* $\sin(n\pi)$*.*

Solution: We have the following table

n	0	1	2	3	4	5	...
$\sin(n\pi)$	0	0	0	0	0	0	...

$\Rightarrow \sin(n\pi) = 0$

(c) *Write out* $\sin\left(\frac{n\pi}{2}\right)$*,* $n = 0, 1, 2, \ldots$ *(There isn't a simpler way to express this.)*

Solution:

n	0	1	2	3	4	5	6	7	...
$\sin(n\pi/2)$	0	1	0	−1	0	1	0	−1	...

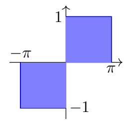
Problem 3. *Compute the Fourier series of the standard, odd, period* 2π *square wave* $\mathrm{sq}(t)$*. Do this by computing the integrals for its Fourier coefficients.*

Solution: Period 2π implies $L = \pi$, so $\dfrac{n\pi}{L} = n$.

Fourier formula: $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \mathrm{sq}(t)\cos(nt)\,dt$. Since $\mathrm{sq}(t)$ is given in cases, we need to break the integral into cases.

$$\begin{aligned}
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} \overbrace{\mathrm{sq}(t)}^{-1}\cos(nt)\,dt + \int_{0}^{\pi} \overbrace{\mathrm{sq}(t)}^{1}\cos(nt)\,dt\right] \\
&= \frac{1}{\pi}\left[-\frac{\sin(nt)}{n}\bigg|_{-\pi}^{0} + \frac{\sin(nt)}{n}\bigg|_{0}^{\pi}\right] \\
&= 0 \quad (\text{since } \sin(0) = \sin(n\pi) = 0).
\end{aligned}$$

Also $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \mathrm{sq}(t)\,dt = \dfrac{1}{\pi}\left[\int_{-\pi}^{0} -1\,dt + \int_{0}^{\pi} 1\,dt\right] = \dfrac{1}{\pi}\left[-\pi + \pi\right] = 0.$

(Could also do this geometrically  area = 0)

Likewise,

$$\begin{aligned} b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} \overbrace{\mathrm{sq}(t)}^{-1} \sin(nt)\,dt + \int_{0}^{\pi} \overbrace{\mathrm{sq}(t)}^{1} \sin(nt)\,dt\right] \\ &= \frac{1}{\pi}\left[\left.\frac{\cos(nt)}{n}\right|_{-\pi}^{0} - \left.\frac{\cos(nt)}{n}\right|_{0}^{\pi}\right] \\ &= \frac{1}{\pi}\left[\frac{1-(-1)^n}{n} - \frac{(-1)^n - 1}{n}\right] \\ &= \frac{1}{\pi}\left[\frac{2-2(-1)^n}{n}\right] = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{n\pi} & \text{if } n \text{ odd} \end{cases} \end{aligned}$$

Summary $a_0 = 0, \quad a_n = 0, \quad b_n = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{n\pi} & \text{if } n \text{ odd} \end{cases}$

Thus,

$$\begin{aligned} \mathrm{sq}(t) &= \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + ... + b_1\sin(t) + b_2\sin(2t) + ... \\ &= \frac{4}{\pi}\sin(t) + \frac{4}{3\pi}\sin(3t) + \frac{4}{5\pi}\sin(5t) + ... \\ &= \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + ...\right) \\ &= \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt)}{n} \qquad \textbf{You should learn this by heart.} \end{aligned}$$