Day 44, F 4/12/2024

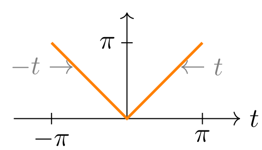
Topic 21: Fourier series (day 2)
Jeremy Orloff

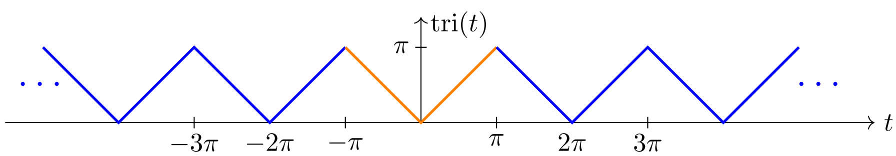
1 Agenda

- Finish yesterday's notes
- Fourier series of tri(t)
- Fourier approximation applet
- Decay rate of coefficients – heuristics

2 Standard triangle wave

(Also in yesterday's notes.)

tri(t): period 2π; over $-\pi$ to π: $\text{tri}(t) = |t| =$ 



Example 1. Compute the Fourier series of tri(t).

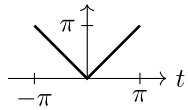
Solution: Period $= 2\pi$, so $L = \pi \longrightarrow \dfrac{n\pi}{L} = n$.

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \text{tri}(t)\cos(nt)\,dt \quad \text{(need to split into cases)} \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -t\,\cos(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi} t\,\cos(nt)\,dt \\
&= \ldots \text{ (by parts or table lookup)} \\
&= \begin{cases} -\frac{4}{n^2\pi} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}
\end{aligned}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \text{tri}(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -t\,dt + \frac{1}{\pi}\int_{0}^{\pi} t\,dt = \pi.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \text{tri}(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -t\,\sin(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi} t\,\sin(nt)\,dt = \ldots = 0.$$

So, $\boxed{\text{tri}(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}}$.

Note: we could have computed a_0 as $\frac{1}{\pi} \cdot \text{area}\Big($ 

$\Big) = \pi$.

3 Fourier approximation applet

https://web.mit.edu/jorloff/www/OCW-ES1803/fourierapproximation.html

4 Decay rates of coefficients (in Topic 22)

When a sequence goes to 0, we can talk about how fast it decays.

- $1/n$ goes to 0 as $n \to \infty$. We say it decays like $1/n$.
- $1/n^2$ goes to 0 faster than $1/n$. We say it decays like $1/n^2$.

We only care about the rough rate of decay. So we say $3/n$ decays like $1/n$ and $\frac{1}{n^2+n}$ decays like $1/n^2$.

For series: the faster the coefficients decay, the faster the series converges.

Decay rate of Fourier coefficients

1. If $f(t)$ has a jump, then at least one of a_n or b_n decay like $1/n$. **Example:** $\text{sq}(t)$.

2. If $f(t)$ has a corner, then at least one of a_n or b_n decay like $1/n^2$. **Example:** $\text{tri}(t)$.

3. If $f(t)$ is smooth, then a_n, b_n decay like $1/n^3$ or faster. The smoother the function, the faster the decay.