Day 45, T 4/16/2024

Topic 22: Fourier series
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1 Agenda

- Decay rates of Fourier coefficients
- Convergence of Fourier series
- Applet: Gibbs' phenomenon
- Calculation tricks: even and odd functions
- Finite series

2 Decay rates of Fourier coefficients

See in-class notes for the previous class or the Topic 22 notes.

3 Convergence of Fourier series

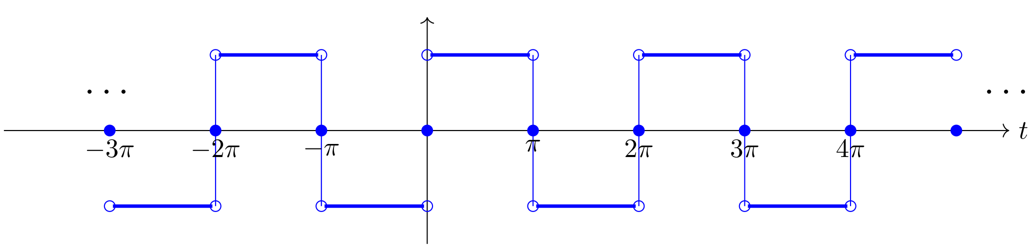
See Topic 22 notes for details: We assume $f(t)$ is differentiable except, possibly, for some jump discontinuities.

- If $f(t)$ is periodic, its Fourier series converges to $f(t)$ at points t where it is continuous.
- Its Fourier series converges to the midpoint of the gap at jump discontinuties.

Example 1. sq(t) is not defined at the jump discontinuities. Its Fourier series is

$$\text{sq}(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}.$$

This converges to the function graphed below.



For DEs, this is not very important. We don't usually need to worry about the value at the jump.

3.1 Gibbs' phenomenon at jump discontinuities

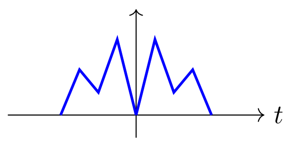
We will talk about this in class. It is discussed in the Topic 22 notes. A careful proof is given in an enrichment note posted with the topic notes.

https://web.mit.edu/jorloff/www/OCW-ES1803/fourierapproximation.html

(Look at the zoomed in part of the square wave and discontinuous sawtooth.)

4 Even and odd functions

$f(t)$ is even if $f(-t) = f(t) \longleftrightarrow$ symmetric across the y-axis.

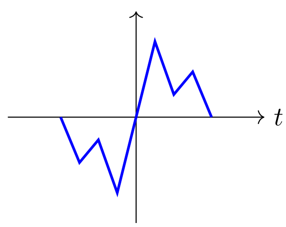


Even function

Examples: 1, t^2, t^4, ... , $\cos(t)$.

Integrals: By symmetry $\displaystyle\int_{-L}^{L} f(t)\,dt = 2\int_0^L f(t)\,dt$.

$f(t)$ is odd if $f(-t) = -f(t) \longleftrightarrow$ symmetric through the origin.



Odd function

Examples: t, t^3, t^5, ... , $\sin(t)$.

Integrals: By symmetry $\displaystyle\int_{-L}^{L} f(t)\,dt = 0$.

4.1 Arithmetic

even·even	=	even	e.g.,	$t^2 \cdot t^4 = t^6$
even·odd	=	odd	e.g.,	$t^2 \cdot t^3 = t^5$
odd·odd	=	even	e.g.,	$t^3 \cdot t^5 = t^8$

4.2 Application to Fourier coefficients

$$f(t): \quad \text{even period } 2L \quad \longrightarrow \quad a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi}{L}t\right)\,dt, \quad b_n = 0$$

$$f(t): \quad \text{odd period } 2L \quad \longrightarrow \quad b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi}{L}t\right)\,dt, \quad a_n = 0$$

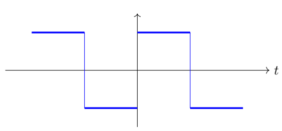
Proof:

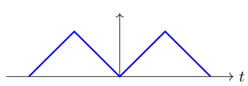
$$f(t)\ \text{even}: \quad a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\overbrace{f(t)}^{\text{even}}\ \overbrace{\cos\left(\frac{n\pi}{L}t\right)}^{\text{even}}}_{\text{even}}\,dt = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi}{L}t\right)\,dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\overbrace{f(t)}^{\text{even}}\ \overbrace{\sin\left(\frac{n\pi}{L}t\right)}^{\text{odd}}}_{\text{odd}}\,dt = 0$$

$f(t)$ odd is similar.

Conclusion: An even periodic function has only cosine terms in its Fourier series. An odd periodic function has only sine terms.

Example 2. $\text{sq}(t) =$  is odd $= \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\sin(nt)}{n}$ (only sine terms).

Example 3. $\text{tri}(t) =$  is even $= \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\cos(nt)}{n^2}$ (only cosine terms).

Note: The sum over n odd has nothing to do with whether the function is even or odd. It's just a feature of these particular functions.

5 Finite series

Example 4. $f(t) = 2\sin(t) + \sin(3t)$

Fourier coefficients: $a_n = 0, \quad b_1 = 2, \quad b_2 = 1, \quad b_3 = b_4 = ... = 0.$

Both a_n and b_n decay like 0.