Solutions Day 45, T 4/16/2024
Topic 22: Fourier series (continued)
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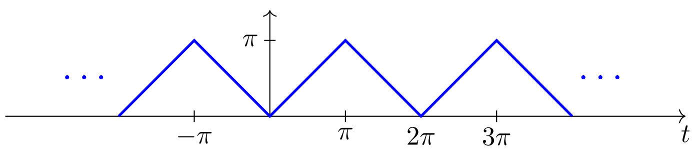
Note: There is a useful integral table on the last page.

Problem 1. *Compute the Fourier series of* tri(t)*, the standard period* 2π *triangle wave. Do this by computing the integrals for its coefficients.*

Solution: Over one period, $\text{tri}(t) = \begin{cases} -t & \text{for } -\pi < t < 0 \\ t & \text{for } 0 < t < \pi \end{cases}$. The half-period $L = \pi$.



Since tri(t) is even, $b_n = 0$. Using the doubling trick for even functions

$$a_n = \frac{2}{\pi}\int_0^\pi \overbrace{\text{tri}(t)}^{t} \cos(nt)\,dt = \frac{2}{\pi}\overbrace{\begin{cases} -2/n^2 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even, } n \neq 0 \end{cases}}^{\text{integral table (2)}}$$

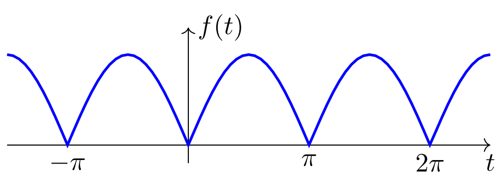
$$a_0 = \frac{2}{\pi}\int_0^\pi \overbrace{\text{tri}(t)}^{t}\,dt = \frac{2}{\pi}\cdot\left.\frac{t^2}{2}\right|_0^\pi = \pi.$$

Summary: $a_0 = \pi,\quad a_n = \begin{cases} -4/\pi n^2 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even, } n \neq 0 \end{cases}.\qquad \text{tri}(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}.$

Problem 2. *Let* $f(t) = |\sin t|$ *(rectified sine curve).*

(a) *Graph this.*

Solution:



(b) *Estimate the decay rate of its Fourier coefficients.*

Solution: The graph has corners, so we will see a decay rate of $1/n^2$ in one of a_n or b_n. Since the function is even, $b_n = 0$. So it is a_n that will decay like $1/n^2$.

(c) *Compute its Fourier series.*

Solution: $f(t)$ has period π, so $L = \pi/2$ and $\frac{n\pi}{L} = 2n$.

Since $f(t)$ is even, $b_n = 0$ and $a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi}{L}t\right)\,dt.$

Using formula 7 in the integral table with $a = 2n$ and $b = 1$ we get

$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2} \sin(t)\cos(2nt)\,dt = \frac{4}{\pi}\int_0^{\pi/2} \sin(t)\cos(2nt)\,dt$$
$$= \frac{4}{\pi}\cdot\frac{1}{2}\left[\frac{-\cos((2n+1)t)}{2n+1} + \frac{\cos((2n-1)t)}{2n-1}\right]\Bigg|_0^{\pi/2}$$

Since $2n+1$ and $2n-1$ are odd, $\cos((2n+1)\pi/2) = 0$, $\cos((2n-1)\pi/2) = 0$. So,

$$a_n = \frac{2}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right] = -\frac{4}{\pi}\cdot\frac{1}{4n^2-1}.$$

$a_0 = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2} \sin t\,dt = \dfrac{4}{\pi}(-\cos t)\Big|_0^{\pi/2} = \dfrac{4}{\pi}.$

So, $f(t) = \underbrace{\dfrac{2}{\pi}}_{a_0/2} - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{\cos(2nt)}{4n^2-1}.$

(d) *Confirm your answer to Part (b).*

Solution: Yes, the coefficients a_n decay like $1/n^2$.

Problem 3. *Say whether each of the following functions is even, odd or neither.*

(a) $t^2\sin(3t)$

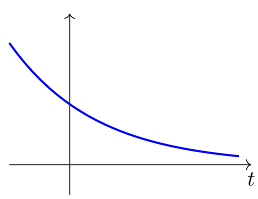
Solution: Even · odd = $\boxed{\text{odd}}$.

(b) $t^2\sin(3t) + t^2\cos(3t)$

Solution: Odd + even = $\boxed{\text{neither}}$.

(c) e^{-t}

Solution: $\boxed{\text{Neither}}$

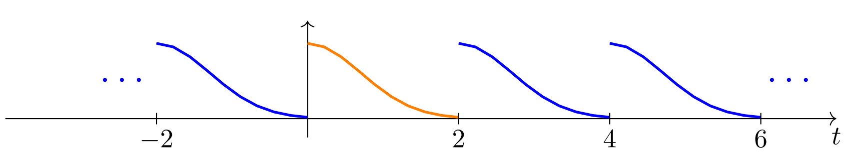


(d) $t\sin(8t)$

Solution: Odd · odd = $\boxed{\text{even}}$

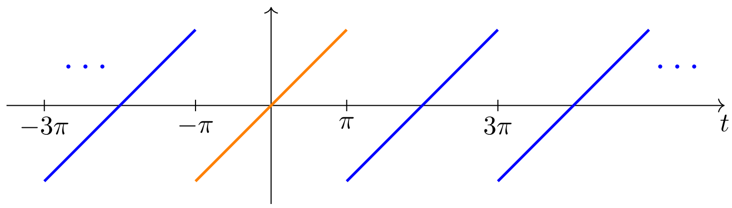
(e) *$f(t)$ has period 2. $f(t) = e^{-t^2}$ for $0 \le t \le 2$.*

Solution: $\boxed{\text{Neither}}$



(f) *$f(t)$ has period 2π; $f(t) = 2\pi t$ for $-\pi < t < \pi$*

Solution: $\boxed{\text{Odd}}$

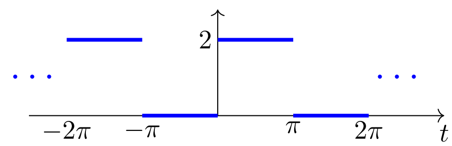


Problem 4. *Let $f(t) = e^{\sin t}$. What is the period of $f(t)$? Estimate the decay rate of its coefficients.*

Solution: Since $\sin t$ has period 2π, so does $f(t)$. Since $f(t)$ has derivatives of all order, the decay rate of its coefficients is really fast –faster than $\frac{1}{n^k}$ for any k.

I graphed $f(t)$ and its Fourier series out to $n = 4$. The match was perfect. Computing numerically, $a_5 \approx -6.88 \times 10^{-11}$ and $b_5 = 5.4 \times 10^{-4}$.

Problem 5. *If didn't do this last class: Let $f(t) = 1 + \text{sq}(t)$*

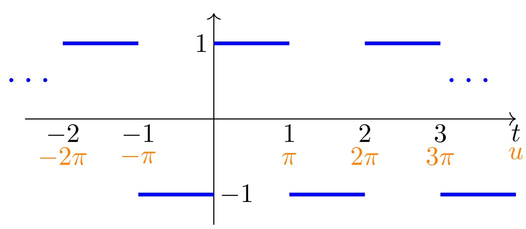


Graph of $f(t)$

Find the Fourier series.

Solution: See solutions for previous class.

Problem 6. *If didn't do this last class: Let $g(t)$ have period 2 and $g(t) = \begin{cases} -1 & \text{for } -1 < t < 0 \\ 1 & \text{for } 0 < t < 1. \end{cases}$*



Graph of $g(t)$

Find the Fourier series for $g(t)$.

Solution: See solutions for previous class.

Integrals *(for n a positive integer)*

1. $\displaystyle\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$

2. $\displaystyle\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$

3. $\displaystyle\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$

4. $\displaystyle\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$

1′. $\displaystyle\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$

2′. $\displaystyle\int_0^\pi t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$

3′. $\displaystyle\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \text{for } n \text{ odd} \\ \dfrac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases}$

4′. $\displaystyle\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$

If $a \neq b$

5. $\displaystyle\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

6. $\displaystyle\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

7. $\displaystyle\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$

8. $\displaystyle\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$

9. $\displaystyle\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$

10. $\displaystyle\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$