Day 47, R 4/18/2024

Topic 23: Compuation tricks, sine and cosine series

Jeremy Orloff

1 Agenda

- This wil take 2 days
- Problems from yesterday
- Calculation tricks: shifting, scaling, differentiation, integration
- Sine and cosine series

2 Calculation shortcuts

Example 1. Let $f(t) = 3 + \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$.

Find Fourier series for $f_a(t) = 2f(t) - 5$, $f_b(t) = f(\pi t)$, $f_c(t) = f'(t)$.

Solution: Just do the same thing to the Fourier series

$f_a(t) = 2\left(3 + \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}\right) - 5 = 1 + 2\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$.

$f_b(t) = 3 + \sum_{n=1}^{\infty} \frac{\cos(n\pi t)}{n^2}$.

$f_c(t) = \left(3 + \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}\right)' = \sum_{n=1}^{\infty} -\frac{\sin(nt)}{n}$.

Example 2. Same $f(t)$. Find a series for $g(t) = \int f(t)\,dt$.

Solution: $g(t) = C + 3t + \sum_{n=1}^{\infty} \frac{\sin(nt)}{n^3}$.

Note: $g(t)$ is not periodic, so this is not a Fourier series. But it is still useful for 1803!

Example 3. Same $f(t)$. Find a series for $h(t) = f(t - \pi/4)$.

Solution:

$$\begin{aligned} h(t) &= 3 + \sum_{n=1}^{\infty} \frac{\cos(n(t-\pi/4))}{n^2} \quad \text{(Not officially a Fourier series, but still useful.)} \\ &= 3 + \sum_{n=1}^{\infty} \frac{\cos(n\pi/4)}{n^2}\cos(nt) + \sum_{n=1}^{\infty} \frac{\sin(n\pi/4)}{n^2}\sin(nt) \quad \text{(Fourier series)} \end{aligned}$$

Example 4. Let $f(t) = \cos(\pi t) + 2\cos(2\pi t) + 3\cos(3\pi t)$.

$f(t)$ is periodic. What is its fundamental angular frequency? Its period? Its Fourier series?

Solution: Every term's frequency is a multiple of the fundamental frequency, so the fundamental frequency is $\omega = \pi$. The base period is $2\pi/\omega = 2$.

Alternatively, the base period must be common to each term. Each term has period 2, so $f(t)$ has base period 2.

The Fourier series is exactly as given, i.e., $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, all other coefficients are 0.

Example 5. Same question for $g(t) = 2\cos(2\pi t) + 3\cos(3\pi t)$.

Solution: The greatest common divisor of the frequencies in the terms is π, so this is the fundamental frequency.

For the base period:

$\cos(2\pi t)$ has period 1, 2, 3, 4, ...

$\cos(3\pi t)$ has period 2/3, 4/3, 2, 8/3, ...

The smallest common period is 2, so $g(t)$ has base period 2.

Alternatively, the base period is $2\pi/\pi = 2$.

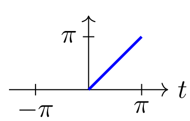
The Fourier series is exactly as given.

Do problems 1, 2

3 Fourier sine and cosine series

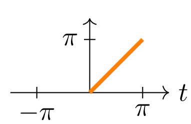
- Semantically different from Fourier series.
- Understand and derive using Fourier series.
- For functions defined on the interval $[0, L]$.
- Will be important in Topic 25. L = physical length.

Example 6. $f(x) = x$ on $[0, \pi]$. (So, $L = \pi$.)

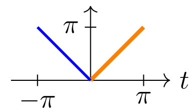


$f(x)$ is not periodic – it is not even defined for all x. So $f(x)$ cannot have a Fourier series.

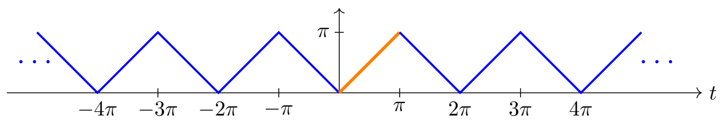
3.1 Periodic extensions

For everything that follows, let's let $f(x) = x$ on $[0, \pi]$ = 

Example 7. Graph the even, period 2π extension of f.

Solution: First mirror $f(x)$ across the y-axis: 

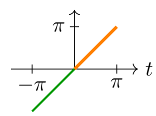
Then extend this to an even, period 2π function



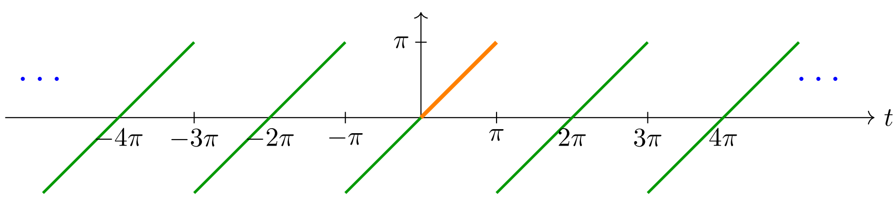
$\tilde{f}_e(x)$ = even period 2π extension

This is called the even, period 2π extension of $f = \tilde{f}_e(f)$.

Example 8. Graph the odd, period 2π extension of f.

Solution: First mirror $f(x)$ across the origin: 

Then extend this to an odd, period 2π function

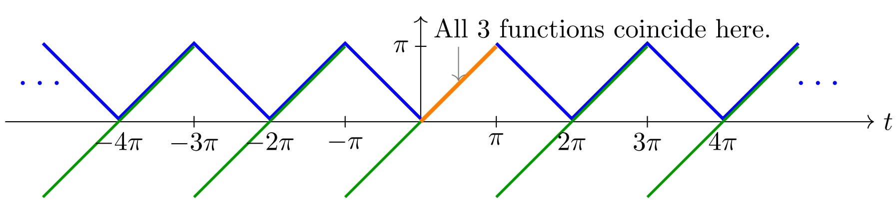


$\tilde{f}_o(x)$ = odd period 2π extension

This is called the odd, period 2π extension of $f = \tilde{f}_o(f)$.

Here are all 3 functions. Notice that on the interval $[0, \pi]$, $\underbrace{f(x) = \tilde{f}_e(x) = \tilde{f}_o(x)}$.

This is why $\tilde{f}_e$, $\tilde{f}_o$ are called extensions of $f(x)$



f and its two extensions coincide over $[0, \pi]$

3.2 Sine and cosine series

We can now use the Fourier series of the periodic extensions to define Fourier cosine and Fourier sine series for $f(x)$.

$\tilde{f}_e(x)$ is even, period π. So,

$$\tilde{f}_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \quad \text{where } a_n = \frac{2}{\pi}\int_0^{\pi} \overbrace{\tilde{f}_e(x)}^{f(x)} \cos(nx)\,dx.$$

Since f and $\tilde{f}_e$ agree on $[0,\pi]$:

On $[0,\pi]$, $\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)}$. This is called the Fourier cosine series for $f(x)$.

Likewise, $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin(nx)\,dx$

– is called the Fourier sine series for $f(x)$

– is the Fourier series of $\tilde{f}_o(x)$

Summary:

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \sum_{n=1}^{\infty} b_n \sin(nx)$.

– These are the Fourier cosine and Fourier sine series for $f(x)$.

– They are valid on the interval $[0,\pi]$.

– Coefficients can be computed using only $f(x)$ (and not the extensions).

– They are also the Fourier series for $\tilde{f}_e(x)$ and $\tilde{f}_o(x)$.

– This works the same for any L (not just π).

Example 9. Always check if the extensions are known functions.

For example, in Example 7, $\tilde{f}_e(x) = \text{tri}(x)$.

So the cosine series is $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nx)}{n^2}$