Solutions Day 47, R 4/18/2024
Topic 23: Computation tricks, sine and cosine series
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Note: There is a useful integral table on the last page.

Problem 1. *Find the Fourier series for each of the following functions.*

(a) $f(t) = 2 + 3\text{sq}(t)$.

Solution: $\text{sq}(t) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt)}{n} \quad \Rightarrow f(t) = 2 + \frac{12}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt)}{n}$.

(b) $g(t) = \text{sq}(\pi t)$.

Solution: $g(t) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\pi t)}{n}$.

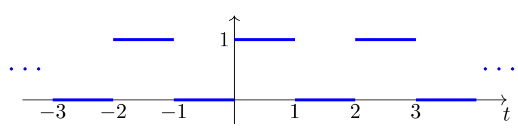
(c) $h(t) = \text{sq}(t - \pi/4)$.

Solution:

$$h(t) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(n(t-\pi/4))}{n} = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin\left(nt - \frac{n\pi}{4}\right)}{n} \quad \text{(not officially a Fourier series, usually fine for 18.03)}$$
$$= \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos\left(\frac{n\pi}{4}\right)\sin(nt)}{n} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin\left(\frac{n\pi}{4}\right)\cos(nt)}{n} \quad \text{(official Fourier series).}$$

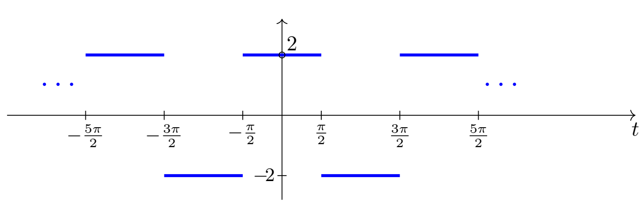
Problem 2. *Find the Fourier series for each of the following functions.*

(a)



Solution: The function is $\frac{1}{2}(1 + \text{sq}(\pi t)) = \frac{1}{2} + \frac{2}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\pi t)}{n}$.

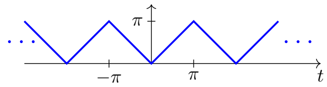
(b)



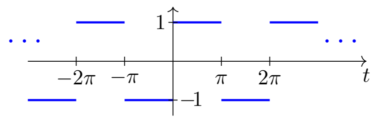
Solution: The function is $\underbrace{2\,\text{sq}\left(t + \frac{\pi}{2}\right)}_{\text{shifted left}} = \underbrace{\frac{8}{\pi}\sum_{n \text{ odd}} \frac{\sin\left(nt + \frac{n\pi}{2}\right)}{n}}_{\text{Usually a good enough answer}}$

So, function $= \underbrace{\frac{8}{\pi}\left(\cos(t) - \frac{\cos(3t)}{3} + \frac{\cos(5t)}{5} - ...\right)}_{\text{Fourier series}}$

Problem 3. *Show* $\frac{d}{dt}\text{tri}(t) = \text{sq}(t)$*. Show the same is true for their Fourier series.*

Solution: $\text{tri}(t) =$ 

The slopes of the lines in $\text{tri}(t)$ are ± 1. So,

$\text{tri}'(t) =$  $= \text{sq}(t).$

Or: $\text{tri}(t) = \begin{cases} -t & \text{for } -\pi < t < 0 \\ t & \text{for } 0 < t < \pi \end{cases}$, $\text{tri}(t)$ has period 2π.

Being careful! No jump discontinuities in $\text{tri}(t)$. So, over 1 period,

$$\text{tri}'(t) = \text{ regular derivative } = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi. \end{cases} = \text{sq}(t) \quad \blacksquare$$

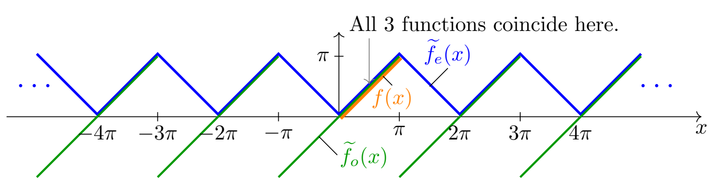
Fourier series:

$$\text{tri}(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt)}{n^2} \quad \Rightarrow \text{tri}'(t) = -\frac{4}{\pi} \sum_{n \text{ odd}} \frac{-n\sin(nt)}{n^2} = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n} = \text{sq}(t) \quad \blacksquare$$

Problem 4. *Let* $f(x) = x$ *on the interval* $[0, \pi]$.

(a) *Graph* $f(x)$ *over this interval.*

Solution: Here are the graphs for the problem. $f(x)$ in orange, just on the interval $[0, \pi]$. $\tilde{f}_e(x)$ in blue (triangle wave), $\tilde{f}_o(x)$ in green (discontintuous sawtooth).



(b) *Extend the graph to an even, period* 2π *function.*

Call this function $\tilde{f}_e$*. Find its Fourier series.*

Solution: The graph is above. Since $\tilde{f}_e(x) = \text{tri}(x)$, we know its Fourier series. $\tilde{f}_e(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nx)}{n^2}$.

(c) *Repeat Part (b) for the odd, period* 2π *extension* $\tilde{f}_o(x)$.

Hint: $\frac{2}{\pi} \int_0^\pi x \sin(nx)\, dx = \frac{-2(-1)^n}{n}$.

Solution: This is an odd period 2π function. So, $a_n = 0$ and

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx)\, dx = \frac{2}{\pi} \int_0^\pi x \sin(nx)\, dx = \frac{-2(-1)^n}{n}. \text{ (Thanks helpful hint!)}$$

Thus, $\tilde{f}_o(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$.

(d) *Give the Fourier sine and cosine series for $f(x)$.*

Solution: These are the same as the Fourier series for $\tilde{f}_o$ and $\tilde{f}_e$.

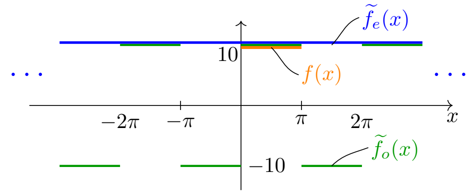
$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nx)}{n^2} \qquad \underbrace{\text{valid on } 0 \le x \le \pi.}_{\text{Key point}}$$

Problem 5. *Let $f(x) = 10$ on $0 \le x \le \pi$.*

Find the Fourier sine and cosine series for $f(x)$.

Hint: no computation needed.

Solution: We graph $f(x)$ and its even and odd 2π extensions.



We see that $\tilde{f}_e(x) = 10$ (constant function). This is its Fourier series. So the Fourier cosine series for $f(x) = 10$.

We see that $\tilde{f}_o(x) = 10\,\text{sq}(x) = \frac{40}{\pi}\sum_{n \text{ odd}} \frac{\sin(nx)}{n} =$ Fourier sine series.

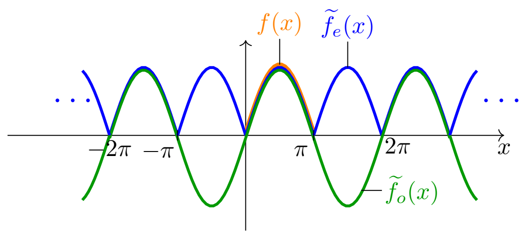
Problem 6. *Let $L = \pi$, $\quad f(x) = \sin x$ on $\quad 0 \le x \le \pi$.*

Find the Fourier sine and cosine series for f.

Don't actually compute the integrals. Just name them and use the name.

Discuss the decay rate of the coefficients for each series.

Solution: It is not strictly necessary to plot the even and odd period 2π extensions. We do it anyway because it often helps us predict the decay rate of coefficients or avoid computing integrals.



$\tilde{f}_e(x)$ has corners (it's called a rectified sine curve), so the coefficients should decay like $\frac{1}{n^2}$.

$L = \pi$, so $\dfrac{n\pi}{\pi} = n$

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin x\,dx = \frac{4}{\pi}.$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x\cos(nx)\,dx = \frac{4}{\pi}\begin{cases} -\frac{1}{n^2-1} & n \text{ even} \\ 0 & n \text{ odd}\end{cases} \qquad \text{(Found using the integral table)}$$

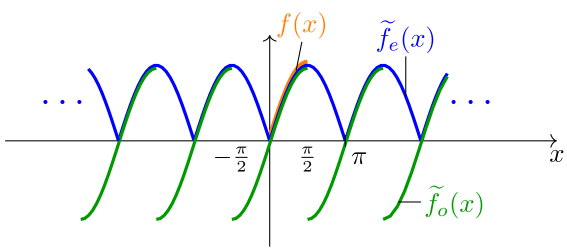
So the cosine series for $\sin x$ on $0 \le x \le \pi$ is

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n \text{ even}} \frac{\cos(nx)}{n^2-1} = \frac{2}{\pi} - \frac{4}{\pi}\cdot\frac{\cos(2x)}{3} - \frac{4}{\pi}\cdot\frac{\cos(4x)}{15} - \frac{4}{\pi}\cdot\frac{\cos(6x)}{35} - \ldots$$

$\tilde{f}_o(x) = \sin x$, which is already written as a Fourier series. So the sine series for $\sin x$ on $0 \le x \le \pi$ is $\sin x$. The coefficients decay like 0.

Problem 7. *Let $L = \pi/2$, $f(x) = \sin x$ on $0 \le x \le \pi/2$. Repeat the previous problem.*

Solution: We graph the period $2L = \pi$ extensions:



Cosine series: $\tilde{f}_e(x)$ has corners, so the coefficients decay like $\frac{1}{n^2}$.

$L = \frac{\pi}{2} \quad \Rightarrow \frac{n\pi}{L} = 2n.$

$$a_0 = \frac{4}{\pi}\int_0^{\pi/2} \sin x\,dx = \frac{4}{\pi}.$$

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} \sin x\,\cos(2nx)\,dx = -\frac{4}{\pi}\cdot\frac{1}{4n^2-1}. \quad \text{(Looked up in integral table.)}$$

So, on $0 \le x \le \pi/2$,

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^\infty \frac{\cos(2nx)}{4n^2-1} = \frac{2}{\pi} - \frac{4}{3\pi}\cos(2x) - \frac{4}{15\pi}\cos(4x) - \ldots$$

Note: It happens that $\tilde{f}_e(x)$ is the rectified sine curve, so this is the same cosine series as in the previous problem.

Sine series: $\tilde{f}_o(x)$ has jumps, so the coefficients decay like $\frac{1}{n}$.

$$b_n = \frac{4}{\pi}\int_0^{\pi/2} \sin x\sin(2nx)\,dx = (-1)^{n+1}\frac{8}{\pi}\cdot\frac{n}{4n^2-1} \quad \text{(Found using the integral table.)}$$

Thus, $\tilde{f}_o(x) = \dfrac{8}{\pi}\displaystyle\sum_{n=1}^\infty (-1)^{n+1}\cdot\frac{n}{4n^2-1}\sin(2nx) =$ Fourier sine series for $\sin x$ on $\underbrace{0 < x < \pi/2}_{\text{Key point}}$.

Integrals *(for n a positive integer)*

1. $\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$

2. $\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$

3. $\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$

4. $\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$

1′. $\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$

2′. $\int_0^\pi t\cos(nt)\,dt = \begin{cases} \frac{-2}{n^2} & \textit{for } n \textit{ odd} \\ 0 & \textit{for } n \neq 0 \textit{ even} \end{cases}$

3′. $\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \frac{\pi^2}{n} - \frac{4}{n^3} & \textit{for } n \textit{ odd} \\ \frac{-\pi^2}{n} & \textit{for } n \neq 0 \textit{ even} \end{cases}$

4′. $\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$

If $a \neq b$

5. $\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

6. $\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

7. $\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$

8. $\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$

9. $\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$

10. $\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$