Problems Day 47, R 4/18/2024

Topic 23: Computation tricks, sine and cosine series

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Note: There is a useful integral table on the last page.

Problem 1. Find the Fourier series for each of the following functions.

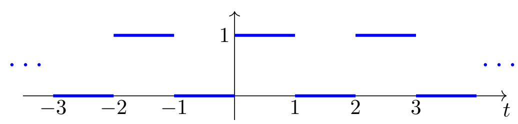
(a) $f(t) = 2 + 3\text{sq}(t)$.

(b) $g(t) = \text{sq}(\pi t)$.

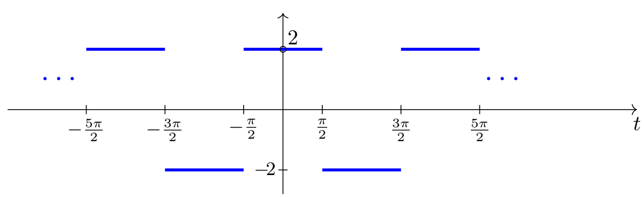
(c) $h(t) = \text{sq}(t - \pi/4)$.

Problem 2. Find the Fourier series for each of the following functions.

(a)



(b)



Problem 3. Show $\dfrac{d}{dt}\text{tri}(t) = \text{sq}(t)$. Show the same is true for their Fourier series.

Problem 4. Let $f(x) = x$ on the interval $[0, \pi]$.

(a) Graph $f(x)$ over this interval.

(b) Extend the graph to an even, period 2π function.

Call this function $\tilde{f}_e$. Find its Fourier series.

(c) Repeat Part (b) for the odd, period 2π extension $\tilde{f}_o(x)$.

Hint: $\dfrac{2}{\pi}\displaystyle\int_0^\pi x\sin(nx)\,dx = \dfrac{-2(-1)^n}{n}$.

(d) Give the Fourier sine and cosine series for $f(x)$.

Problem 5. Let $f(x) = 10$ on $0 \le x \le \pi$.

Find the Fourier sine and cosine series for $f(x)$.

Hint: no computation needed.

Problem 6. Let $L = \pi$, $\quad f(x) = \sin x$ on $\quad 0 \le x \le \pi$.

Find the Fourier sine and cosine series for f.

Don't actually compute the integrals. Just name them and use the name.

Discuss the decay rate of the coefficients for each series.

Problem 7. Let $L = \pi/2$, $f(x) = \sin x$ on $0 \le x \le \pi/2$. Repeat the previous problem.

Integrals (for n a positive integer)

1. $$\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$$

$1'$. $$\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$$

2. $$\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$$

$2'$. $$\int_0^\pi t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$$

3. $$\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$$

$3'$. $$\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \text{for } n \text{ odd} \\ \dfrac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases}$$

4. $$\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$$

$4'$. $$\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$$

If $a \neq b$

5. $$\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$

6. $$\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$

7. $$\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$$

8. $$\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$$

9. $$\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$$

10. $$\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$$