Day 49, M 4/22/2024

Topic 24: $P(D)x =$ **periodic**
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1 Agenda

- DEs with periodic input
- Resonance and near resonance

2 DEs with periodic input

Example 1. Solve $x' + 3x = \text{sq}(t)$.

Solution: Use Fourier series and superposition.

Written out, the DE is

$$x' + 3x = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t + \overbrace{\ldots}^{\text{important}}$$

Solve one term at a time: $x_n' + 3x_n = \dfrac{4}{n\pi}\sin(nt)$.

SRF: $x_{n,p}(t) = \dfrac{4}{n\pi}\dfrac{\sin(nt - \phi(n))}{|P(i\,n)|}$

$P(in) = 3 + i\,n \longrightarrow |P(i\,n)| = \sqrt{9+n^2}$, $\boxed{\phi(n) = \text{Arg}(P(i\,n)) = \tan^{-1}(n/3) \text{ in Q1}}$

So, $x_{n,p}(t) = \dfrac{4\sin(nt - \phi(n))}{\pi\, n\sqrt{9+n^2}}$. Thus,

$$\begin{aligned} x_p &= x_{1,p} + x_{3,p} + \ldots = \frac{4}{\pi}\frac{\sin(t-\phi(1))}{\sqrt{10}} + \frac{4}{\pi}\frac{\sin(3t-\phi(3))}{3\sqrt{18}} + \frac{4}{\pi}\frac{\sin(5t-\phi(5))}{5\sqrt{34}} + \ldots \\ &= \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(nt-\phi(n))}{n\sqrt{9+n^2}}. \end{aligned}$$

3 Resonance and near resonance (review)

Example 2. Consider $2x'' + 0.1x' + 18x = \cos(t) + \cos(2t) + \cos(3t)$.

The natural frequency of the system is $\sqrt{18/2} = 3$.

Lightly damped $\longrightarrow$ resonance frequency is near 3.

The input term $\cos(3t)$ is near the resonance frequency. So $\cos(3t)$ causes a near resonant respsonse.

Example 3. Solve $x'' + 9x = \cos(3t)$.

Solution: $P(3i) = 0$. So the input frequency is the pure resonance frequency. The extended SRF gives, $x_p(t) = \dfrac{t\cos(3t-\phi)}{|P'(3i)|}$, where $\phi = \text{Arg}(P'(3i))$.

$P'(r) = 2r$, so $P(3i) = 6i = 6e^{i\pi/2}$. Thus, $x_p(t) = \dfrac{t\cos(3t-\pi/2)}{6}$.

Pure resonance $\longrightarrow$ response amplitude is "∞".