Solutions Day 49, M 4/22/2024

Topic 24: $P(D)x =$ periodic

Jeremy Orloff

Problem 1. *Solve $\ddot{x} + 9x = \text{sq}(t)$*

– Watch for resonance.

– Use the notation $\phi(n)$ or ϕ_n in the SRF.

Solution: The input $\text{sq}(t) = \dfrac{4}{\pi}\displaystyle\sum_{n \text{ odd}} \dfrac{\sin(nt)}{n}$ is a superposition. So we solve in pieces:

$$\ddot{x}_n + 9x_n = \sin(nt) \qquad \text{(We'll bring the coefficients in later.)}$$

Characteristic equation: $P(r) = r^2 + 9$.

For $n \neq 3$, $P(in) = 9 - n^2$, $\Rightarrow |P(in)| = |9 - n^2|$, $\phi(n) = \text{Arg}(P(in)) = \begin{cases} 0 & n < 3 \\ \pi & n > 3 \end{cases}$.

So, using the SRF, for $n \neq 3$, $x_{n,p}(t) = \dfrac{\sin(nt - \phi(n))}{|P(in)|} = \dfrac{\sin(nt - \phi(n))}{|9 - n^2|}$

For $n = 3$, $P(3i) = 0$. So we'll need $P'(r) = 2r$.

$$P'(3i) = 6i \quad \Rightarrow |P'(3i)| = 6, \quad \text{Arg}(P'(3i)) = \frac{\pi}{2}.$$

Using the extended SRF, $x_{3,p}(t) = \dfrac{t\sin(3t - \pi/2)}{6}$.

Using superposition:

$$x_p(t) = \sum_{n \text{ odd}} \frac{4}{n\pi} x_{n,p}(t) = \frac{4\sin(t - \phi(1))}{8\pi} + \frac{4t\sin(3t - \pi/2)}{18\pi} + \sum_{n \text{ odd},\ n \geq 5} \frac{4\sin(nt - \phi(n))}{\pi n\sqrt{9 - n^2}}$$

$$= \frac{4\sin(t)}{8\pi} + \frac{4t\sin(3t - \pi/2)}{18\pi} + \sum_{n \text{ odd},\ n \geq 5} \frac{4\sin(nt - \pi)}{\pi n\sqrt{9 - n^2}}.$$

There is pure resonance due to the $n = 3$ term.

Problem 2. *Solve $\ddot{x} + 16x = \text{sq}(t)$. Is there any resonance?*

Solution: This is similar to the previous problem. The system has a resonant frequency at $\omega = 4$. Since $\omega = 4$ is not a frequency in (the Fourier series of) $\text{sq}(t)$, we don't need to worry about resonance.

Pieces: $\ddot{x}_n + 16x_n = \sin(nt)$

$$\text{SRF (for } n \neq 4\text{)}, \quad x_{n,p}(t) = \frac{\sin(nt - \phi(n))}{|16 - n^2|}, \qquad \phi(n) = \begin{cases} 0 & n < 4 \\ \pi & n > 4 \end{cases}.$$

So, $x_p(t) = \dfrac{4}{\pi}\displaystyle\sum_{n \text{ odd}} \dfrac{x_{n,p}(t)}{n} = \dfrac{4}{\pi}\displaystyle\sum_{n \text{ odd}} \dfrac{\sin(nt - \phi(n))}{n|16 - n^2|}$.

Problem 3. *Solve* $\ddot{x} + 0.01\dot{x} + 9x = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$.

Are there any near-resonant terms?

Solution: Pieces: $\ddot{x}_n + 0.01\dot{x}_n + 9x_n = \cos(nt)$.

$P(in) = 9 - n^2 + 0.01\, n\, i.$ So,

$$|P(in)| = \sqrt{(9-n^2)^2 + (0.01 \cdot n)^2}, \quad \boxed{\phi(n) = \mathrm{Arg}(P(in)) = \tan^{-1}\left(\frac{0.01 \cdot n}{9 - n^2}\right) \text{ in Q1, Q2}}.$$

So, $x_{n,p}(t) = \frac{\cos(nt - \phi(n))}{|P(in)|} = \frac{\cos(nt - \phi(n))}{\sqrt{(9-n^2)^2 + (0.01 \cdot n)^2}}$

Thus, $x_p(t) = \sum_{n=1}^{\infty} \frac{x_{n,p}(t)}{n^2} = \sum_{n=1}^{\infty} \frac{\cos(nt - \phi(n))}{n^2\sqrt{(9-n^2)^2 + (0.01 \cdot n)^2}}$.

The undamped system, $\ddot{x} + 9x = \cos(\omega t)$ has pure resonance at $\omega = 3$. So, the lightly damped system, $\ddot{x} + 0.01\dot{x} + 9x = \cos(\omega t)$ will have practical resonance at some ω near 3. Since $\omega = 3$ is a frequency in the given input, the response to this input will have a near resonant term. The amplitude of the $n = 3$ term in the response is $\frac{1}{3^2 \cdot (0.03)} = \frac{100}{27}$.

Problem 4. *Solve* $\ddot{x} + 10x = \mathrm{sq}(t)$.

Are there any near-resonant terms?

Solution: Pieces: $\ddot{x}_n + 10x_n = \sin(nt)$.

$$P(in) = 10 - n^2 \quad \Rightarrow |P(in)| = |10 - n^2|, \quad \phi(n) = \mathrm{Arg}(P(in)) = \begin{cases} 0 & n < \sqrt{10} \\ \pi & n > \sqrt{10} \end{cases}.$$

So, $x_{n,p} = \frac{\sin(nt - \phi(n))}{|10 - n^2|}$.

Thus, $x_p(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{x_{n,p}(t)}{n} = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt - \phi(n))}{n|10 - n^2|}$.

The natural frequency of this system is $\sqrt{10}$. Since 3 is close to $\sqrt{10}$, the $n = 3$ term in $\mathrm{sq}(t)$ is a near-resonant term.