Day 5, F 2/9/2024

Topic 3: Input response

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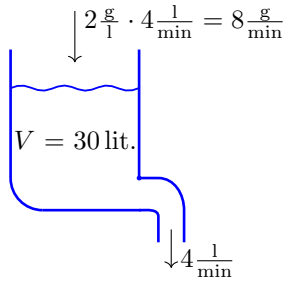
1 Agenda

- Read Topic 4 notes for tomorrow (super-important topic)
- Mixing tank example
- Engineering language: system-input-response
- A useful format: $x(t) = x_0 e^{a(t-t_0)}$

2 Example: mixing tanks

Suppose we have a tank with 30 liters of water.

Say a salt solution with concentration 2 g/l is added at the rate of 4 l/min. At the same time, solution is drawn out of the tank at 4 l/min.



Model the amount of salt in the tank over time.

Solution: Note: The tank is balanced, i.e., volume stays constant.

Let $x(t)$ be the amount of salt in the tank in grams.

$$\begin{aligned} \frac{dx}{dt} = \text{ rate of change } &= \text{rate in} - \text{rate out} \\ &= 8\,\frac{\text{g}}{\text{min}} - 4\,\frac{\text{l}}{\text{min}} \cdot \frac{x(t)\,\text{grams}}{V\,\text{liters}} \\ &= 8 - \frac{4x}{30} \end{aligned}$$

So, $x' = 8 - \dfrac{4}{30}\,x$ or $x' + \dfrac{4}{30}\,x = 8$ (first-order linear).

If we solved this, we would find $x(t) = 60 + Ce^{-4t/30}$.

3 Engineering language

- Signal: a function of time. It carries information.
- System: a mechanism for converting an input signal to an output signal.
- Output or response: the output signal from a system

In general, it is up to the engineer to say what is called the input and output. That is, there is no formal mathematical definition of the input and output.

In 1803, the output signal is usually the solution to a DE modeling the system. We'll use an example to talk about input.

Example 1. (Annealing steel)

Simplify by assuming Newton's law of cooling holds.

$T(t)$ = temperature of the steel

$E(t)$ = temperature of the annealing bath = the annealing schedule

Natural to call $E(t)$ the input, i.e., this is what the engineer gives the technician.

Newton: $T' = -k(T - E(t)) \quad \Leftrightarrow \quad \underbrace{T' + kT = kE(t)}_{\text{system}}$

$T(t)$ = output or response.

Notice: The right-hand side of the equation is $kE(t)$. The k is not part of the input.

3.1 Mathematical input

When we have a DE with no physical system in mind, e.g., $x' + 3x = f(t)$, we will often borrow the engineering language and call $f(t)$ the "mathematical input". Often, we shorten this to just "input".

When we have our engineering hats on, we need to be more careful with what we designate as the input.

4 Useful format for solutions to $x' = ax$

The DE $x' = ax$ with initial condition $x(t_0) = x_0$ has solution

$$x(t) = x_0 e^{a(t-t_0)}.$$

This is often easier to work with than writing the solution as Ce^{at} and solving for C:

$$x(t_0) = x_0 = Ce^{at_0} \quad \Rightarrow\ C = x_0 e^{-at_0} \quad \Rightarrow\ x(t) = x_0 e^{-at_0} e^{at},$$

Which is the same answer in a slightly different form.

ES.1803 Differential Equations

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