Solutions Day 5, F 2/9/2024

Topic 3: Input response

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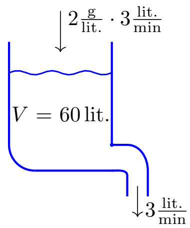
Problem 1. *We have a mixing tank. Initially it contains 60 l of pure H_2O. We add a salt solution with a concentration of 2 g/l at a rate of 3 l/min. At the same time, we draw off solution at 3 l/min.*

Draw a picture and then model the amount of salt in the tank. After doing the other problems come back and solve the DE.

Solution: Let $x(t)$ be the amount of salt in the tank.



x' = rate in − rate out $= 6\dfrac{\text{g}}{\text{min}} - 3\dfrac{\text{l}}{\text{min}} \cdot \dfrac{x(t)}{V}\dfrac{\text{g}}{\text{l}}$.

V = constant = 60.

So, $x' = 6 - \dfrac{x}{20}$ or $x' + \dfrac{1}{20}x = 6$, $x(0) = 0$.

Solve the DE (first-order linear) using the variation of parameters formula.

The homogeneous solution is $x_h(t) = e^{-\int \frac{1}{20}\,dt} = e^{-t/20}$. So,

$$x(t) = x_h(t)\int \frac{6}{x_h(t)} + Cx_h(t) = e^{-t/20}\int 6e^{t/20}\,dt + Ce^{-t/20}.$$

So, $x(t) = e^{-t/20}\left(120e^{t/20}\right) + Ce^{-t/20} = 120 + Ce^{-t/20}$.

Initial condition: $x(0) = 0 = 120 + C \quad \Rightarrow C = -120$.

So, $\boxed{x(t) = 120 - 120e^{-t/20}}$.

Important note: Don't become too wedded to using variation of parameters. For constant coefficient equations like this one, we will learn easier and faster techniques.

Problem 2. *Show the DE $x'' + 8x' + 7x = f(t)$ is linear, i.e., satisfies the superposition principle.*

Solution: Assume $x_1'' + 8x_1' + 7x_1 = f_1(t)$ and $x_2'' + 8x_2' + 7x_2 = f_2(t)$.

We have to show that, for any constants c_1, c_2,

$$x = c_1x_1 + c_2x_2 \text{ satisfies } x'' + 8x' + 7x = c_1f_1 + c_2f_2. \qquad (*)$$

To do this, we plug x into $(*)$.

$$\begin{aligned} x'' + 8x' + 7x &= (c_1x_1 + c_2x_2)'' + 8(c_1x_1 + c_2x_2)' + 7(c_1x_1 + c_2x_2) \\ &= c_1 \underbrace{(x_1'' + 8x_1' + 7x_1)}_{f_1} + c_2 \underbrace{(x_2'' + 8x_2' + 7x_2)}_{f_2} \quad \text{(algebra)} \\ &= c_1 f_1 + c_2 f_2 \quad \text{(by hypothesis)} \quad \blacksquare \end{aligned}$$

Problem 3. *Show that the DE* $y' + y^2 = q(t)$ *does not satisfy the superposition principle. ($q(t)$= "input" can be any function.)*

Solution: We have to check the superposition principle starting from its hypotheses:

$$\text{Suppose} \quad \begin{cases} y_1' + y_1^2 = q_1, \text{ i.e., } y_1 \text{ solves the DE } y' + y^2 = q_1 \quad \text{and} \\ y_2' + y_2^2 = q_2, \text{ i.e., } y_2 \text{ solves the DE } y' + y^2 = q_2 \end{cases}.$$

Now check if $y = c_1y_1 + c_2y_2$ solves $y' + y^2 = c_1q_1 + c_2q_2$ (c_1, c_2 constants)

Substituting y into the DE

$$\begin{aligned} y' + y^2 &= (c_1y_1 + c_2y_2)' + (c_1y_1 + c_2y_2)^2 \\ &= c_1y_1' + c_2y_2' + c_1^2y_1^2 + 2c_1c_2y_1y_2 + c_2^2y_2^2. \end{aligned} \qquad (*)$$

The terms with c_1^2, c_2^2 and c_1c_2 make it pretty clear that this is not $c_1q_1 + c_2q_2$. We can show this formally by looking at specific cases. It only requires one special case, where linearity fails, to show the DE is not linear. For good measure, we'll give two cases.

Case 1. Take $c_1 = 2$, $c_2 = 0$. Then $(*)$ becomes

$$y' + y^2 = 2y_1' + 4y_1^2 = 2(y_1 + y_1^2) + 2y_1^2 = 2q_1 + 2y_1^2 \neq 2q_1. \quad \blacksquare$$

Case 2. Take $c_1 = 1$, $c_2 = 1$. Then $(*)$ becomes

$$\begin{aligned} y' + y^2 &= y_1' + y_2' + y_1^2 + 2y_1y_2 + y_2^2 \\ &= y_1 + y_1^2 + y_2 + y_2^2 + 2y_1y_2 \\ &= q_1 + q_2 + 2y_1y_2 \\ &\neq q_1 + q_2. \end{aligned}$$

That is, for $y = y_1 + y_2$, $y' + y^2 \neq q_1 + q_2$. $\blacksquare$

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