Day 50, T 4/23/2024

Topic 25: PDEs (day 1 of 2)
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1 Agenda

- Heat and wave equations (partial differential equations)
- PDE models – linearity, superposition
- Boundary conditions – linearity, superposition
- Fourier method of separation of variables
- General solution (∞ parameters)
- Initial conditions (used to determine values for parameters)

2 Preliminary: functions of 2 independent variables

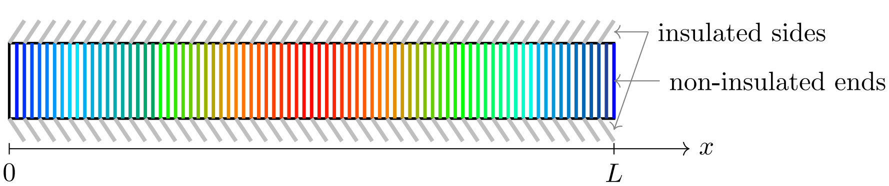
$u(x,t)$:	x, t	independent
	$0 \le x \le L$,	x = physical variable, e.g., position
	$t \ge 0$	t = time

Notation: $\dfrac{\partial u}{\partial t} = u_t$, $\dfrac{\partial^2 u}{\partial t^2} = u_{tt}$, $\dfrac{\partial u}{\partial x} = u_x$ etc.

3 Heat equation

This models the temperature of a heated bar

- The temperature can be different at different points.
- The temperature changes in time.
- The sides are insulated so heat can only enter or leave through the ends.

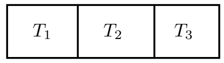


$u(x,t)$ = temperature at position x at time t.

L = length of the bar, $0 \le x \le L$, $t \ge 0$.

Heat equation: $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$, $k > 0$ is a physical constant.

We will see solutions like $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx)$ (and variations on this theme).

This model is more realistic than 

, i.e., dividing the bar into a finite number of sections each with a uniform temperature.

4 Important consequence of independence of x and t

(We will need this soon.)

Suppose x and t are independent variables and $g(x)$ is a function of x and $h(t)$ is a function of t.

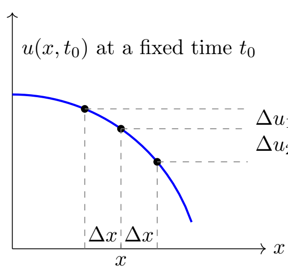
If $g(x) = h(t)$ for all (x, t), then both $g(x)$ and $h(t)$ are constant functions.

Proof: Fix $t = 0$ and let x vary. Then $g(x) = h(0)$ for all x, i.e., $g(x)$ is constant. Likewise, $h(t)$ is constant.

5 Handwaving justification of why the heat equation is a reasonable model

(Probably won't do in class.)

Consider a fixed time t. Plot $u(x, t)$ at this time.



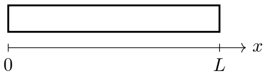
If the curve is concave down, then $\Delta u_2 > \Delta u_1$. This means that more heat will flow from x to the cooler section to its right than will flow to x from the warmer section to its left, i.e., $\left.\frac{\partial u}{\partial t}\right|_{(x,t)} < 0$. Since the curve is concave down $\frac{\partial^2 u}{\partial x^2} < 0$, so the equation $\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$ has the correct sign.

6 Linearity (superposition principle)

The heat equation is linear and homogeneous. So it satisfies the superposition principle: If u_1, u_2 are solutions, then so is $u = c_1 u_1 + c_2 u_2$ for any constants c_1, c_2. –Easy to check.

Note: Writing the DE as $\frac{\partial u}{\partial t} - k\,\frac{\partial^2 u}{\partial x^2} = 0$ makes it more obviously homogeneous.

7 Boundary conditions (BC)

For our heated bar 

, $x = 0$, $x = L$ are the boundary of the bar.

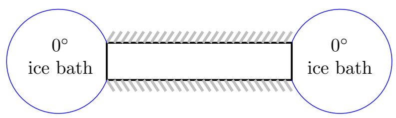
Example 1. Boundary conditions (BC)

Suppose the ends of the bar are maintained at 0°, e.g., they are placed in ice baths. This means $u(0,t) = 0$ and $u(L,t) = 0$ for all time t. These are called boundary conditions.

8 Solving using Fourier's method of separation of variables

Example 2. Suppose $L = \pi$ and we have the following PDE (heat equation) with boundary conditions.

PDE: $u_t = 5u_{xx}, \quad 0 \le x \le \pi, \quad t \ge 0$
BC: $u(0,t) = 0, \quad u(\pi,t) = 0.$



(a) Find all the separated solutions to the PDE.

(b) From the solutions in Part (a), find those that also satisfy the boundary conditions. (These are called modal solutions.)

(c) Use superposition to give the general solution to the PDE and BC

Solution: (a) (Method of optimism) Try $u(x,t) = X(x)T(t)$ (separated solution).

Plug in: $\dfrac{\partial u}{\partial t} = X(x)T'(t)$, $\dfrac{\partial^2 u}{\partial x^2} = X''(x)T(t)$. So,

$$\text{PDE:} \quad X(x)T'(t) = 5X''(x)T(t).$$

Separate the variables:

$$\underbrace{\frac{T'(t)}{5T(t)}}_{\text{Convention: put the coefficient here}} = \frac{X''(x)}{X(t)} \overbrace{= \text{ constant}}^{\text{Function of } t \,=\, \text{function of } x} = \overbrace{-\lambda}^{\text{Convention: use it!}}$$

A little algebra leads to two linear, constant coefficient homogeneous DEs:

$$\begin{aligned} T' &= -5\lambda T &&\longrightarrow T' + 5\lambda T = 0 \\ X'' &= -\lambda X &&\longrightarrow X'' + \lambda X = 0 \end{aligned}$$

Here, λ is any constant, but the same in both equations.

$X'' + \lambda X$ has characteristic equation $r^2 + \lambda = 0$.

The roots are $\pm\sqrt{-\lambda} \quad \longrightarrow$ 3 cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

Case $\lambda > 0$:

Pure imaginary roots: $\pm\sqrt{\lambda}\,i \quad \longrightarrow X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$.

Solve $T' + 5\lambda T = 0$ for this case: $\quad T(t) = ce^{-5\lambda t}$.

Separated solution to PDE: $\quad u(x,t) = X(x)T(t) = \underbrace{\left(a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)\right)e^{-5\lambda t}}_{\text{Dropped } c\text{. It's redundant.}}$.

Case $\lambda = 0$:

Repeated roots: 0, 0 $\quad \longrightarrow X(x) = a + bt$

Solve $T' = 0$ for this case: $\quad T(t) = c$.

Separated solution to PDE: $\quad u(x,t) = X(x)T(t) = a + bt$. (Dropped c. It's redundant.)

Case $\lambda < 0$:

Real roots: $\pm\sqrt{-\lambda} \quad \longrightarrow X(x) = ae^{\sqrt{-\lambda}\,x} + be^{-\sqrt{-\lambda}\,x}$.

Solve $T' + 5\lambda T = 0$: $\quad T(t) = ce^{-5\lambda t}$.

Separated solution to PDE: $\quad u(x,t) = X(x)T(t) = \underbrace{\left(ae^{\sqrt{-\lambda}\,x} + be^{-\sqrt{-\lambda}\,x}\right)e^{-5\lambda t}}_{\text{Dropped } c\text{. It's redundant.}}$.

Lots of separated solutions with parameters a, b λ.

(b) Modal solutions = separated solutions that match boundary conditions.

For a separated solution $u(x,t) = X(x)T(t)$, the boundary condition $u(0,t) = 0$ becomes

$$X(0)T(t) = 0 \quad \longrightarrow X(0) = 0 \text{ or } T(t) = 0.$$

If $T(t) = 0$, then $u(x,t) = 0$, i.e., $u(x,t)$ is the trivial solution. Since we want nontrivial solutions, the boundary condition becomes $X(0) = 0$. Likewise, the other boundary condition is $X(\pi) = 0$.

Checking the boundary conditions for each case:

Case $\lambda > 0$: $\quad$ We have $X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$.

The BC give $\quad X(0) = a = 0, \quad X(\pi) = a\cos(\sqrt{\lambda}\,\pi) + b\sin(\sqrt{\lambda}\,\pi) = 0$.

Since $a = 0$, the second condition is $b\sin(\sqrt{\lambda}\,\pi) = 0 \;\Rightarrow\;$ either $b = 0$ or $\sin(\sqrt{\lambda}\,\pi) = 0$.

If $b = 0$, then $X(x) = 0$, i.e., we have a trivial solution.

If $\sin(\sqrt{\lambda}\,\pi) = 0$, $\quad$ then $\quad \sqrt{\lambda} = 1, 2, 3, \ldots$.

Thus, when $\lambda > 0$, the nontrivial $X(x)$ are $b\sin(nx)$, for $n = 1, 2, 3, \ldots$. So, in this case, we have the following (nontrivial) modal solutions:

$$u_n(x,t) = b_n \sin(nx)\, e^{-5n^2 t}, \qquad \text{where, } n = 1,\, 2,\, 3,\, \ldots$$

(We use the index n to distinguish the modal solutions from each other.)

Case $\lambda = 0$: $\quad$ We have $X(x) = a + bx$.

BC: $\quad X(0) = a = 0, \quad X(\pi) = a + b\pi$.

The only solution to these equations is $a = 0$, $b = 0$. That is, in the case $\lambda = 0$, there are only trivial solutions satisfying the boundary conditions.

Case $\lambda < 0$: We have $X(x) = ae^{\sqrt{-\lambda}\,x} + be^{-\sqrt{-\lambda}\,x}$.

A small bit of algebra (see the topic notes) will show there are only trivial solutions satisfying the boundary conditions.

With our conventions, the case $\lambda < 0$ will always only produce trivial solutions.

We have found all the modal solutions:

$$u_n(x,t) = b_n \sin(nx)e^{-5n^2t}, \ n = 1, 2, 3, \ldots$$

(We carefully index the function and coefficient, so they have names.)

(c) Since **both** the PDE and BC are linear and homogeneous, we can superposition the modal solutions.

The general solution to the PDE which also satisfies the BC is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-5n^2t}.$$

Tomorrow: Initial conditions, wave equation.