Solutions Day 50, T 4/23/2024
Topic 25: PDES (2 days)
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Problem 1. *Consider the heat equation with boundary conditions:*

PDE: $u_t = 3u_{xx}, \quad 0 \le x \le 1, \quad t \ge 0$

BC: $u(0,t) = 0, \quad u(1,t) = 0$

(a) *Find all the separated solutions to the PDE.*

Solution: Method of optimism (Fourier separation of variables): Try $u(x,t) = X(x)T(t)$.

Plug in: $u_t = 3u_{xx} \quad \Rightarrow X(x)T'(t) = 3X''(x)T(t)$.

Separate variables: $\dfrac{T'(t)}{3T(t)} = \dfrac{X''(x)}{X(x)} \overbrace{= \text{ constant}}^{\text{function of } t \,=\, \text{function of } x} = \overbrace{-\lambda}^{\text{convention} - \text{use it!}}$.

Algebra: $T' = -3\lambda T, \quad X'' = -\lambda X, \quad \lambda$ any constant (same in both equations). So we have the two DEs:

$$X'' + \lambda X = 0, \qquad T' + 3\lambda T = 0.$$

The characteristic roots depend on the cases: $\lambda > 0, \quad \lambda = 0, \quad \lambda < 0$. We go through the cases one at a time.

<u>Case $\lambda > 0$</u>.

$X'' + \lambda X = 0 \quad \Rightarrow \quad \text{roots } r = \pm i\sqrt{\lambda} \quad \Rightarrow X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$.

$T' + 3\lambda T = 0 \quad \Rightarrow \quad \text{roots } r = -3\lambda \quad \Rightarrow T(t) = c\,e^{-3\lambda t}$.

Thus, $u(x,t) = X(x)T(t) = \left(a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)\right)e^{-3\lambda t}$. (Dropped c –it's redundant.)

<u>Case $\lambda = 0$</u>.

$X'' = 0 \quad \Rightarrow X(x) = ax + b$.

$T' = 0 \quad \Rightarrow T = c$.

Thus, $u(x,t) = X(x)T(t) = a + bx$. (Dropped c –it's redundant.)

<u>Case $\lambda < 0$</u>.

$X'' + \lambda X = 0 \quad \Rightarrow \quad \text{roots } r = \pm\sqrt{-\lambda}$ (these are real) $\quad \Rightarrow X(x) = ae^{\sqrt{-\lambda}\,x} + be^{-\sqrt{-\lambda}\,x}$.

$T + 3\lambda T = 0 \quad \Rightarrow T(t) = ce^{-3\lambda t}$. (Same as case $\lambda > 0$.)

Thus, $u(x,t) = X(x)T(t) = \left(ae^{\sqrt{-\lambda}\,x} + be^{-\sqrt{-\lambda}\,x}\right)e^{-3\lambda t}$. (Dropped c –it's redundant)

Lots of separated solutions!

(b) *Find all the modal solutions, i.e., separated solutions to the PDE also satisfying the boundary conditions.*

Solution: Modal solutions = separated solutions that match the boundary conditions.

Checking the boundary conditions: For a separated solution $u(x,t) = X(x)T(t)$, the boundary condition $u(0,t) = X(0)T(t) = 0$ becomes

$$X(0)T(t) = 0 \quad \longrightarrow X(0) = 0 \text{ or } T(t) = 0.$$

If $T(t) = 0$, then $u(x,t) = 0$, i.e., $u(x,t)$ is the trivial solution. Since we are looking for nontrivial solutions, we need $X(0) = 0$.

Likewise, the boundary condition $u(1,t) = 0$ requires $X(1) = 0$ for a nontrivial solution.

In summary, when checking the BC, we can ignore $T(t)$.

Case $\lambda > 0$: We have $X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$.

The BC are $X(0) = a = 0, \quad X(1) = a\cos(\sqrt{\lambda}) + b\sin(\sqrt{\lambda}) = 0$.

Since $a = 0$, the second condition becomes $b\sin(\sqrt{\lambda}) = 0 \quad \Rightarrow b = 0$ or $\sin(\sqrt{\lambda}) = 0$.

If $b = 0$, then $X(x) = 0$, i.e., we have a trivial solution.

If $\sin(\sqrt{\lambda}) = 0$, then $\sqrt{\lambda} = \pi, 2\pi, 3\pi, ...$

We've found the following modal solutions when $\lambda > 0$ $(T(t) = ce^{-3\lambda t} = ce^{-3n^2\pi^2 t})$:

$$u_n(x,t) = b_n \sin(n\pi x)\, e^{-3n^2\pi^2 t}, \qquad \text{where}, n = 1,\ 2,\ 3,\ ...$$

(We use the index n to distinguish the modal solutions from each other.)

Case $\lambda = 0$: We have $X(x) = a + bx$.

The BC are $X(0) = a = 0, \quad X(1) = a + b = 0$.

The only solution is $a = 0$, $b = 0$, i.e., there are only trivial solutions in this case.

(Case $\lambda < 0$): We have $u(x,t) = (ae^{\sqrt{-\lambda}\,x} + be^{-\sqrt{-\lambda}\,x})$.

It is not hard to check that this case has only trivial solutions that match the BC.

In fact, for all problems, the case $\lambda < 0$ will only have trivial solutions matching the BC. So, in future problems, we will ignore this case.

(c) *Give the general solution to the PDE which also satisfies the BC.*

Solution: Since both the PDE and BC are homogeneous, the general solution to the PDE and BC is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-3n^2\pi^2 t}.$$

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