Day 53, F 4/26/2024

Topic 26: PDEs (day 2 of 2)
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1 Agenda

- Finish Topic 25 day 1
- Initial conditions
- Wave equation
- Applets

2 Finish Topic 25 Day 1

Linearity, heat equation, boundary condition, modal solutions

3 Initial conditions (IC)

Example 1. Suppose $u(x,t)$ satisfies all of the following:

PDE:	$u_t = 5u_{xx}, \quad 0 \le x \le \pi, \quad t \ge 0$
BC:	$u(0,t) = 0, \quad u(\pi, t) = 0$
IC:	$u(x,0) = x$ (initial conditions)

Find $u(x,t)$.

Solution: From Topic 25 Day 1, we know the general solution to the PDE which also satisfies the BC is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-5n^2 t}.$$

Using the initial conditions (IC), (plug in $t = 0$):

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) = x, \quad \text{for } 0 \le x \le \pi.$$

That is, $\sum_{n=1}^{\infty} b_n \sin(nx)$ is the Fourier sine series for x on $[0, \pi]$.

So, $b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx = \frac{2(-1)^{n-1}}{n}$ (integration by parts or table lookup).

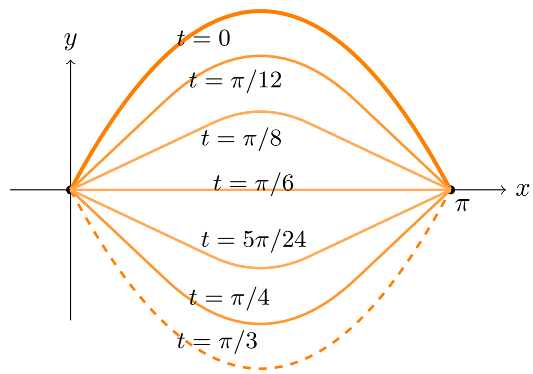
Answer: $u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n} \sin(nx) e^{-5n^2 t}$.

4 Wave equation (models a vibrating string or wire)

- L = length of string.
- Assumes each point on the string vibrates straight up and down.
- Assumes the tension in the string is constant.

These assumptions are approximately true when the amplitude of the displacement is small compared to the length of the string.

Let $y(x,t)$ = displacement at position x at time t.



Shape of the string at various values of t

We model this with the wave equation:

PDE $\quad y_{tt} = a^2 y_{xx};\quad a$ = physical constant with dimension of speed.
BC $\quad$ Similar to heat equation
IC $\quad \underbrace{y(x,0) = f(x)}_{\text{initial position}},\quad \underbrace{y_t(x,0) = g(x)}_{\text{initial velocity}}$ (Need two IC because second-order in t)

Method of solving is the same as for the heat equation

Step 1: Find separated solutions to the PDE

Step 2: Find modal solutions (Separated solutions also satisfying the BC).

Step 3: General solution = superposition of modal solutions.

Step 4: Use the IC to determine the values of the parameters.

In Step 1, separation of variables leads to two ODE;s

$$\underbrace{X''(x) + \lambda X(x) = 0}_{\text{identical to heat equation}},\quad T'' + a^2\lambda T(t) = 0.$$

5 Summary

Heat equation: $\quad u_t = k u_{xx},\quad 0 \le x \le L,\quad t \ge 0.$

Wave equation: $\quad y_{tt} = a^2 y_{xx},\quad 0 \le x \le L,\quad t \ge 0.$

Boundary: $\quad x = 0,\quad x = L.$

Boundary conditions: e.g., $\quad u(0,t) = 0,\quad u(L,t) = 0.$

6 Applets

https://mathlets.org/mathlets/damped-wave-equation/

https://mathlets.org/mathlets/wave-equation/

https://mathlets.org/mathlets/heat-equation/