Solutions Day 53, F 4/26/2024

Topic 26: PDES (day 2)

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Note: There is a useful integral table on the last page.

Problem 1. *Solve the heat equation with insulated ends:*

PDE: $u_t = 5u_{xx}, \quad 0 \le x \le 1, \quad t \ge 0$

BC: $u_x(0,t) = 0, \quad u_x(1,t) = 0$ *(Note derivatives)*

Find the general solution.

Solution: Step 1. Separated solutions: Guess $u(x,t) = X(x)T(t)$.

Plug into PDE: $X(x)T'(t) = 5X''(x)T(t)$.

$$\xrightarrow{\text{algebra}} \frac{T'(t)}{5T(t)} = \frac{X''(x)}{X(x)} = \text{ constant } = -\lambda.$$

So we have two ODEs: $X'' + \lambda X = 0, \qquad T' + 5\lambda T = 0.$

Break into cases.

Case $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$; $T(t) = ce^{-5\lambda t}$.

So, $u(x,t) = X(x)T(t) = e^{-5\lambda t}\left(a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)\right)$. (Dropped c, since it's redundant.)

Case $\lambda = 0$: $X(x) = a + bx, \quad T(t) = c$

So, $u(x,t) = X(x)T(t) = a + bx$. (Dropped c, since it's redundant.)

Case $\lambda < 0$: Can ignore this case. It only produces trivial modal solutions.

Step 2. Modal solutions (separated solutions that also satisfy the BC)

For separated solutions, the BC are $X'(0) = 0, \quad X'(1) = 0$.

Case $\lambda > 0$. BC: $X'(0) = \sqrt{\lambda}\,b = 0, \; X'(1) = -\sqrt{\lambda}\,a\sin(\sqrt{\lambda}) + \sqrt{\lambda}\,b\cos(\sqrt{\lambda})$.

Since $b = 0$, the second condition becomes $-\sqrt{\lambda}\,a\sin(\sqrt{\lambda}) = 0 \quad \Rightarrow \; a = 0$ or $\sin(\sqrt{\lambda}) = 0$.

If $a = 0$, then $X(x) = 0$ and we have a trivial solution.

If $\sin(\sqrt{\lambda}) = 0$, then $\sqrt{\lambda} = \pi, 2\pi, 3\pi, \ldots = n\pi$, where $n = 1, 2, 3, \ldots$.

So we have modal solutions

$$u_n(x,t) = a_n\cos(n\pi\,x)e^{-5n^2\pi^2 t}, \quad n = 1, 2, 3, \ldots$$

Case $\lambda = 0$. BC: $X'(0) = b, \quad X'(1) = b$.

So, $X(x) = a$. We now have one more modal solution, which we write as

$$u_0(x,t) = \frac{a_0}{2}.$$

Case $\lambda < 0$. Ignore.

Step 3. Superposition: General solution to the PDE which satisfies the BC

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) e^{-5n^2\pi^2 t}.$$

Problem 2. *Same equation as Problem 1. Use the initial condition (IC)*

$$u(x,0) = x \quad \text{for } 0 \le x \le 1,$$

to determine the values of the coefficients in the solution.

Solution: From Problem 1 we have $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) e^{-5n^2\pi^2 t}$.

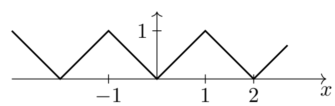
Set $t = 0$: $\quad u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = x, \quad$ on $0 \le x \le 1$.

This is a Fourier cosine series for x (with $L = 1$). So,

$$a_n = 2\int_0^1 x\cos(n\pi x)\,dx = \begin{cases} \frac{-4}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even, } n \ne 0. \end{cases} \qquad \text{(Done using the integral table.)}$$

$$a_0 = 2\int_0^1 x\,dx = 1.$$

So, $\quad u(x,t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{\cos(n\pi x)}{n^2} e^{-5n^2\pi^2 t}.$

Note: The even period 2 extension of x is the triangle wave 

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So we could have found the coefficients without integration by noting that the even extension $= \frac{1}{\pi}\text{tri}(\pi t)$ and using the known Fourier series for $\text{tri}(t)$.

Problem 3. *Discuss the solution to Problem 2*

(a) *in the medium term;*

(b) *in the long-term;*

Solution: Writing out a few terms:

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi x)e^{-5\pi^2 t} - \frac{4}{9\pi^2}\cos(3\pi x)e^{-45\pi^2 t} - \frac{4}{25\pi^2}\cos(5\pi x)e^{-125\pi^2 t} + \ldots$$

(a) In the medium term, the exponentials for $n = 3, 5, 7, \ldots$ go to 0 much much faster than the exponential for $n = 1$. So,

$$u(x,t) \approx \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi x)e^{-5\pi^2 t} \quad \text{(medium term).}$$

(b) In the long-term all the exponentials go to 0. So, $u(x,t) \longrightarrow \frac{1}{2}$.

That is, in an inuslated rod, the temperature becomes constant over time.

Problem 4.
(a) *Solve the wave equation with the given boundary conditions:*

PDE: $y_{tt} = 4y_{xx}, \quad 0 \le x \le \pi, \quad t \ge 0$

BC: $y(0,t) = 0, \quad y(\pi,t) = 0$

Solution: Step 1. Guess a separated solution: $y(x,t) = X(x)T(t)$.

Plug into PDE: $X(x)T''(t) = 4X''(x)T(t) \xrightarrow{\text{algebra}} \dfrac{T''(t)}{4T(t)} = \dfrac{X''(x)}{X(x)} = \text{constant} = -\lambda$.

So we have two ODEs: $\underbrace{X'' + \lambda X = 0}_{\text{same as always}}, \quad \underbrace{T'' + 4\lambda T = 0}_{\text{2nd order}}$.

Break into cases and solve the ODES

Case $\lambda > 0$:

$$\left.\begin{aligned} X(x) &= a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x) \\ T(t) &= c\cos(2\sqrt{\lambda}\,t) + d\sin(2\sqrt{\lambda}\,t) \end{aligned}\right\} \quad y(x,t) = X(x)T(t).$$

Case $\lambda = 0$:

$$\left.\begin{aligned} X(x) &= a + bx \\ T(t) &= c + dt \end{aligned}\right\} \quad y(x,t) = X(x)T(t).$$

Case $\lambda < 0$: Can ignore –no nontrivial modal solutions.

Step2. Modal solutions (separated solutions which satisfy the BC)

For separated solutions, the BC are $X(0) = 0, \quad X(\pi) = 0$.

Case $\lambda > 0$. BC: $X(0) = a = 0, \quad X(\pi) = a\cos(\sqrt{\lambda}\,\pi) + b\sin(\sqrt{\lambda}\,\pi) = 0$.

Since $a = 0$, the second condition becomes $b\sin(\sqrt{\lambda}\,\pi) = 0$. The nontrivial solutions have $\sqrt{\lambda} = n$, where $n = 1, 2, 3, \ldots$.

We have found modal solutions

$$y_n(x,t) = \sin(nx)\left(c_n\cos(2nt) + d_n\sin(2nt)\right).$$

(Dropped the coefficient b since it's redundant.)

Case $\lambda = 0$. BC: $X(0) = a = 0, \quad X(\pi) = a + b\pi = 0$.

This has only the trivial solution $a = 0$, $b = 0$. So, it produces no new modal solutions.

Case $\lambda < 0$: Ignoring this case.

Step 3. General solution to the PDE satisfying the BC:

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(c_n\cos(2nt) + d_n\sin(2nt)\right).$$

(b) *Use the IC* $y(x,0) = 1, \quad y_t(x,0) = 0$ *to find the values of the coefficients in your solution to Part (a).*

Solution: Use the initial conditions in the solution from Part (a):

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx) = 1 \quad \text{on } 0 < x < \pi.$$

This is the sine series for 1 on $(0,\pi)$ = Fourier series of sq(t). So,

$$c_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

We have $y_t(x,t) = \sum_{n=1}^{\infty} \sin(nx)\,(-2n\,c_n \sin(2nt) + 2n\,d_n \cos(2nt))$. So,

$$y_t(x,0) = \sum_{n=1}^{\infty} 2n\,d_n \sin(nx) = 0 \quad \text{on } 0 < x < \pi.$$

This is the sine series for 0. So the coefficients $2n\,d_n = 0 \quad \Rightarrow d_n = 0$

Answer: $\boxed{y(x,t) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(nx)\cos(2nt)}$.

Problem 5. *Consider the heat equation with inhomogeneous BC:*

PDE: $u_t = 2u_{xx}$, $0 \le x \le 1$, $t \ge 0$

BC: $u(0,t) = 1$, $u(1,t) = 3$ *(Inhomogeneous)* *(I)*

(a) *Find a particular solution by guessing a steady-state solution.*

Solution: Steady-state means not changing in time, i.e., $u(x,t) = X(x)$ depends only on x.

Plug this into the PDE $u_t = 2u_{xx}$: $\quad 0 = 2X''(x) \quad \Rightarrow X(x) = a + bx$.

Next we have to match the BC:

$$\left.\begin{aligned} u(0,t) &= X(0) = a = 1 \\ u(1,t) &= X(1) = a + b = 3 \end{aligned}\right\} \quad a = 1,\ b = 2 \quad \Rightarrow X(x) = 1 + 2x.$$

Steady-state solution $\quad u_p(x,t) = 1 + 2x$.

(b) *The associated PDE with homogeneous BC is*

H-PDE: $u_t = 2u_{xx}, \quad 0 \le x \le 1, \quad t \ge 0$

H-BC: $u(0,t) = 0, \quad u(1,t) = 0.$

Solve this and combine it with your answer to Part (a) to give the general solution to the inhomogeneous system with from Part (a).

Solution: The same procedure as in the previous problems will give the general homogeneous solution

$$\underbrace{u_h(x,t)}_{h \text{ for homogeneous}} = \sum_{n=1}^{\infty} b_n \sin(n\pi\, x) e^{-2n^2\pi^2 t}.$$

The general solution to (I) is particular + homogeneous, i.e.,

$$u(x,t) = u_p(x,t) + u_h(x,t) = 1 + 2x + \sum_{n=1}^{\infty} b_n \sin(n\pi\, x) e^{-2n^2\pi^2 t}.$$

(You should check that this satisfies the BC.)

Integrals *(for n a positive integer)*

1. $\displaystyle\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$

2. $\displaystyle\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$

3. $\displaystyle\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$

4. $\displaystyle\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$

1′. $\displaystyle\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$

2′. $\displaystyle\int_0^\pi t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \textit{for } n \textit{ odd} \\ 0 & \textit{for } n \neq 0 \textit{ even} \end{cases}$

3′. $\displaystyle\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \textit{for } n \textit{ odd} \\ \dfrac{-\pi^2}{n} & \textit{for } n \neq 0 \textit{ even} \end{cases}$

4′. $\displaystyle\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$

If $a \neq b$

5. $\displaystyle\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

6. $\displaystyle\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

7. $\displaystyle\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$

8. $\displaystyle\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$

9. $\displaystyle\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$

10. $\displaystyle\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$