Problems Day 53, F 4/26/2024

Topic 26: PDES (day 2)

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Note: There is a useful integral table on the last page.

Problem 1. Solve the heat equation with insulated ends:

PDE: $u_t = 5u_{xx}, \quad 0 \le x \le 1, \quad t \ge 0$

BC: $u_x(0,t) = 0, \quad u_x(1,t) = 0$ (Note derivatives)

Find the general solution.

Problem 2. Same equation as Problem 1. Use the initial condition (IC)

$$u(x,0) = x \quad \text{for } 0 \le x \le 1,$$

to determine the values of the coefficients in the solution.

Problem 3. Discuss the solution to Problem 2

(a) in the medium term;

(b) in the long-term;

Problem 4.

(a) Solve the wave equation with the given boundary conditions:

PDE: $y_{tt} = 4y_{xx}, \quad 0 \le x \le \pi, \quad t \ge 0$

BC: $y(0,t) = 0, \quad y(\pi,t) = 0$

(b) Use the IC $y(x,0) = 1, \quad y_t(x,0) = 0$ to find the values of the coefficients in your solution to Part (a).

Problem 5. Consider the heat equation with inhomogeneous BC:

$$\begin{array}{ll} \textbf{PDE:} & u_t = 2u_{xx},\ 0 \le x \le 1,\ t \ge 0 \\ \textbf{BC:} & u(0,t) = 1,\ u(1,t) = 3\ (\text{ Inhomogeneous}) \end{array} \qquad (I)$$

(a) Find a particular solution by guessing a steady-state solution.

(b) The associated PDE with homogeneous BC is

H-PDE: $u_t = 2u_{xx}, \quad 0 \le x \le 1, \quad t \ge 0$

H-BC: $u(0,t) = 0, \quad u(1,t) = 0.$

Solve this and combine it with your answer to Part (a) to give the general solution to the inhomogeneous system with from Part (a).

Integrals (for n a positive integer)

1. $\displaystyle\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$

2. $\displaystyle\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$

3. $\displaystyle\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$

4. $\displaystyle\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$

1′. $\displaystyle\int_0^{\pi} t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$

2′. $\displaystyle\int_0^{\pi} t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$

3′. $\displaystyle\int_0^{\pi} t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \text{for } n \text{ odd} \\ \dfrac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases}$

4′. $\displaystyle\int_0^{\pi} t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$

If $a \neq b$

5. $\displaystyle\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

6. $\displaystyle\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

7. $\displaystyle\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$

8. $\displaystyle\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$

9. $\displaystyle\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$

10. $\displaystyle\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$