Solutions Day 54, M 4/29/2024

Topic 27: Linear phase portraits (day 1)

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Problem 1. *Suppose A has eigenpairs*

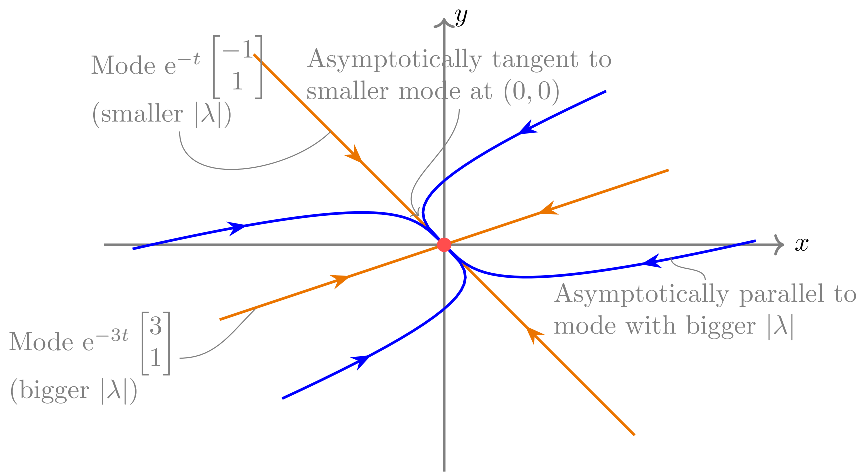
$$\begin{array}{rcll} \lambda & = & -3 & -1 \\ \mathbf{v} & = & \begin{bmatrix} 3 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{array}$$

Sketch a phase portrait of the system $\mathbf{x}' = A\mathbf{x}$. Name the type of critical point at the origin and give its stability.

Solution: We know $\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The negative exponents imply all trajectories go asympotically to 0.

The modes are $c_1 e^{-3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. These have straight line (really rays) trajectories.



Nodal sink; dynamically stable equilibrium at (0,0).

Problem 2. *Suppose A has eigenpairs*

$$\begin{array}{rcll} \lambda & = & -3 & 2 \\ \mathbf{v} & = & \begin{bmatrix} 3 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{array}$$

Sketch a phase portrait of the system $\mathbf{x}' = A\mathbf{x}$. Name the type of critical point at the origin and give its stability.

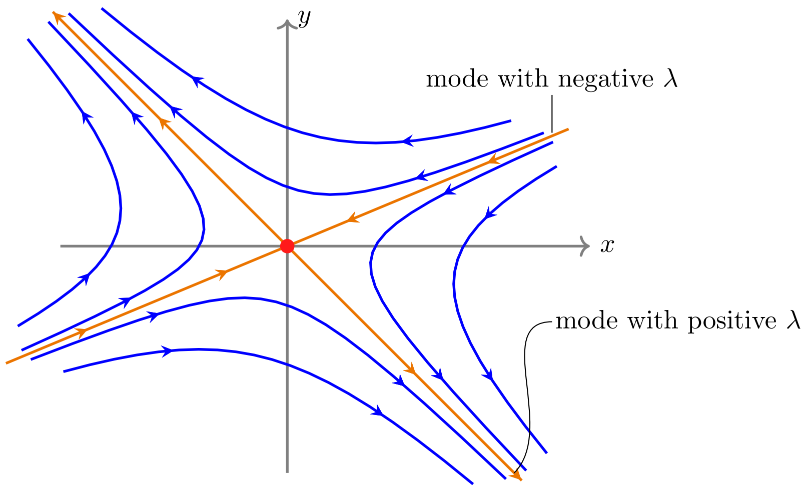
Solution: $\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Modes:

$$\mathbf{x_1} = e^{-3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{goes to 0 as } t \text{ increases}$$

$$\mathbf{x_2} = e^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{goes away from 0 as } t \text{ increases}$$

Mixed model solution, e.g., $e^{-3t}\begin{bmatrix}3\\1\end{bmatrix} + e^{2t}\begin{bmatrix}-1\\1\end{bmatrix}$, goes asympotically to $e^{2t}\begin{bmatrix}3\\1\end{bmatrix}$ as $t \to \infty$ and goes asympotically to $e^{-3t}\begin{bmatrix}-1\\1\end{bmatrix}$ as $t \to -\infty$.



Saddle; dynamically unstable equilibrium as (0,0).