Day 55, T 4/30/2024

Topic 27: Linear phase portraits (day 2)
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1 Agenda

- Phase plane and phase portraits
- Applet: https://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/
- Spirals: sense of turning
- Trace-determinant diagram
- Edge cases

2 Phase plane and phase portraits

- System $\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$
- Trajectory = graph of a solution in the phase plane (xy-plane)
- Always have equilibrium solution: $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is called a critical point because $\mathbf{x}' = A \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

https://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/

3 Spirals: sense of turning

Example 1. Spiral source

Let $\mathbf{x}' = \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix} \mathbf{x} \longrightarrow \lambda = 3 \pm 5i.$

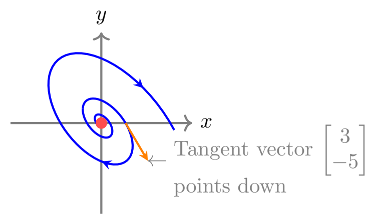
After some algebra: $$\mathbf{x}(t) = c_1 \underset{\substack{\uparrow \\ \text{grows}}}{e^{3t}} \times \underset{\substack{\uparrow \\ \text{circle}}}{\begin{bmatrix} \cos(5t) \\ -\sin(5t) \end{bmatrix}} + c_2 e^{3t} \begin{bmatrix} \sin(5t) \\ \cos(5t) \end{bmatrix}$$

grows $\times$ circle = spiral out

Sense of turning, i.e., clockwise (CW) or counterclockwise (CCW)?

To determine the direction of turning, we look at the point (1,0) in the plane.

$$\begin{aligned} \text{At } (1,0):\ \mathbf{x}' = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} \\ &= \text{ tangent vector to the trajectory through } (1,0) \end{aligned}$$



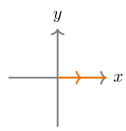
Downward pointing tangent vector at $(1,0)$ $\Rightarrow$ turns clockwise

The tangent vector points down, so the spiral must be turning clockwise.

The critical point at $(0,0)$ is called a spiral source. It is a dynamically unstable equilibrium.

4 Key points about phase portraits

- Trajectories don't cross.
- They fill up the plane.
- Different solutions can have the same trajectory.
  They just have different initial values.
  $\mathbf{x_1} = e^t \begin{bmatrix}1\\0\end{bmatrix}$ has $\mathbf{x_1}(0) = \begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{x_2}(t) = 2e^t \begin{bmatrix}1\\0\end{bmatrix}$ has $\mathbf{x_2}(0) = \begin{bmatrix}2\\0\end{bmatrix}$.

  Both have trajectory 

- Qualitatively, the phase portrait is determined by the eigenvalues.

5 Trace-determinant diagram for 2 by 2 matrices

For $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, the characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix}a-\lambda & b\\ c & d-\lambda\end{vmatrix} = \lambda^2 - \underbrace{(a+d)}_{\text{trace}(A)}\lambda + \underbrace{(ad-bc)}_{\det(A)} = 0.$$

The term $(a+d)$ is called the trace of A, denoted $\text{tr}(A)$. With this notation, the characteristic equation is

$$\lambda^2 - \text{tr}(A)\,\lambda + \det(A) = 0.$$

(Don't forget the minus sign in front of $\text{tr}(A)$.)

Conclusion: The eigenvalues are $\lambda = \dfrac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2}$.

Conclusion: The eigenvalues are determined by $\text{tr}(A)$, $\det(A)$.

Example 2. Let $A = \begin{bmatrix}6 & 5\\1 & 2\end{bmatrix}$. We have, $\text{tr}(A) = 6 + 2 = 8$, $\det(A) = 7$.

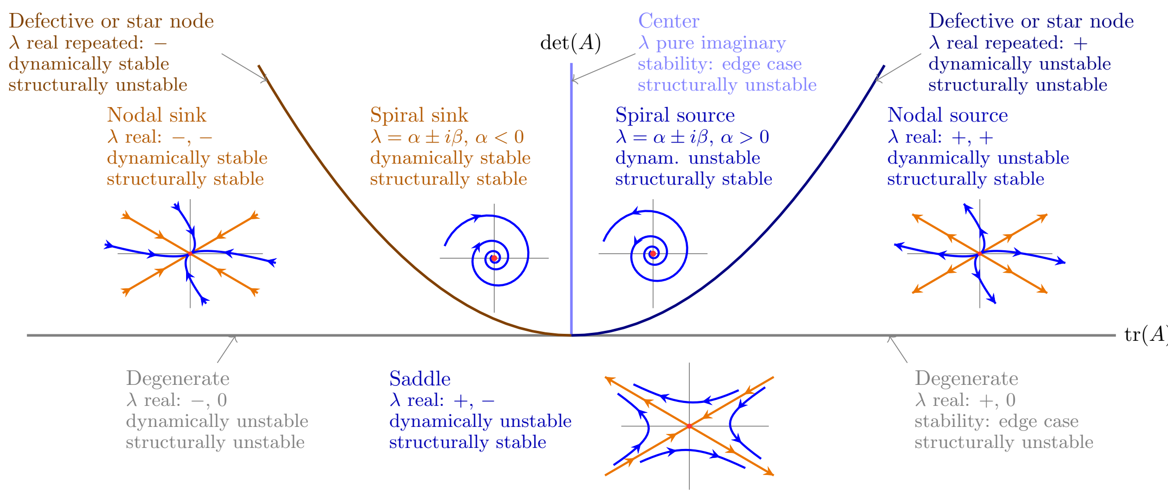
So the characteristic equation is $\lambda^2 - 8\lambda + 7 = 0$.

Example 3. Since $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$ both have trace 8 and determinant 7, they both have the same eigenvalues, $\lambda = 7, 1$.

5.1 Trace and determinnant diagram

$$\lambda = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}.$$

Since the trace and determinant determine the eigenvalues, we can mark out all the cases on the trace-determinant diagram. (The diagram includes all the edge cases, which we will discuss below.)



5.2 All cases

Main types

λ	**Type of critical point**	**Dynamic stability**	**Structural stability**
positive different	nodal source	dynamically unstable	structurally stable
negative different	nodal sink	dynamically stable	structurally stable
one pos., one neg.	saddle	dynamically unstable	structurally stable
complex, pos. real part	spiral source	dynamically unstable	structurally stable
complex, neg. real part	spiral sink	dynamically stable	structurally stable

Edge cases

repeated positive	defective nodal source or star nodal source	dynamically unstable	structurally unstable
repeated negative	defective nodal sink or star nodal sink	dynamically stable	structurally unstable
pure imaginary	center	edge case	structurally unstable
$\lambda_1 = 0$, $\lambda_2 > 0$	degenerate	dynamically unstable	structurally unstable
$\lambda_1 = 0$, $\lambda_2 < 0$	degenerate	edge case	structurally unstable
$\lambda_1 = 0$, $\lambda_2 = 0$	degenerate	??	structurally unstable

Example 4. (Edge case) Plot the phase portrait of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \mathbf{x}$.

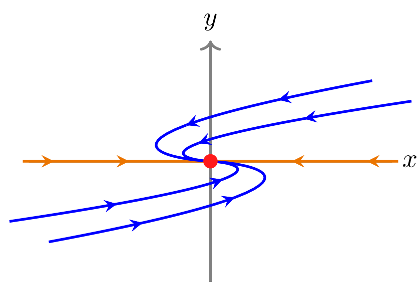
Solution: You can check that $\lambda = -2, -2$, but there is only one independent eigenvector, i.e., this is the defective case.

You can also check the solution is

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left(t e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

This has a defective nodal sink at $(0, 0)$. It is a dynamically stable equilibrium.

There is only one mode, so it must play both roles in the phase portrait. That is, trajectories are tangent to the mode at the origin and asymptotically parallel to it at infinity.



(See the applet.)