Solutions Day 55, T 4/30/2024

Topic 27: Linear phase portraits (day 2+)

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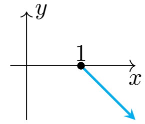
Problem 1. *Let* $A = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$. *Sketch a phase portrait of the system* $\mathbf{x}' = A\mathbf{x}$.

Name the type of critical point at the origin and give its dynamical stability.

Solution: Characteristic equation $\lambda^2 - 6\lambda + 25 = 0 \quad \Rightarrow \lambda = 3 \pm 5i$.

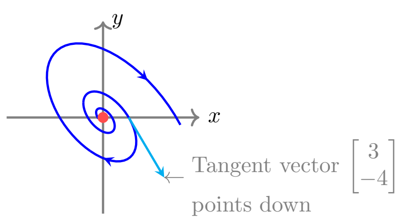
λ complex with positive real part $\quad \Rightarrow$ critical point is a $\boxed{\text{spiral source}}$.

Sense it turns: At $(1,0)$, $\mathbf{x}' = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$



$$\begin{bmatrix} 3 \\ -4 \end{bmatrix} \text{ points down } \longrightarrow \text{ clockwise rotation}$$

Qualititative sketch:



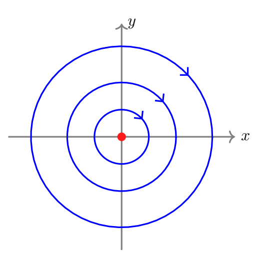
Critical point at (0,0) = spiral source; dynamically unstable equilibrium.

Problem 2. *A system* $\mathbf{x}' = A\mathbf{x}$ *has general solution* $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

Sketch a phase portrait of the system. Name the type of critical point at the origin and give its dynamical stability.

Solution: One solution: $\mathbf{x} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ = circle, clockwise.

Other solutions: $c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ –all circles



Critical point at (0,0) = center. Dynamical stability: Edge case –right between spiral sinks and spiral sources.

Problem 3. *For this problem, we have 2×2 system $\mathbf{x}' = A\mathbf{x}$. For each one, give the type of critical point at $(0, 0)$, its dynamical stability and the structural stability of the system.*

(a) $\mathrm{tr}A = -3, \quad \det A = 2$

(b) $\mathrm{tr}A = -3, \quad \det A = -2$

(c) $\mathrm{tr}A = -2, \quad \det A = 1$

Solution: **(a)** The eigenvalues are $\lambda = \dfrac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2} = \dfrac{-3 \pm \sqrt{9-8}}{2} = -2, \, -1.$

These are negative and distinct. Therefore we have a nodal sink at (0,0): dynamically stable equilibrium, structurally stable system.

(b) $\det A < 0 \quad \Rightarrow$ saddle: dynamically unstable equilibrium, structurally stable system.

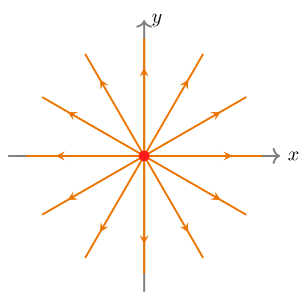
(c) Eigenvalues are $\lambda = \dfrac{-2 \pm \sqrt{4-4}}{2} = -1, \, -1.$

Repeated negative roots: nodal sink (either defective or star): dynamically stable equilibrium, structurally unstable system.

Problem 4. *A system $\mathbf{x}' = A\mathbf{x}$ has solution $\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.*

Sketch a phase portrait.

Solution: Repeated positive eigenvalues. The solution shows this is the complete case. Every solution is a mode.



This is called a star nodal source. The equilibrium is dynamically unstable. It is a structurally unstable system.

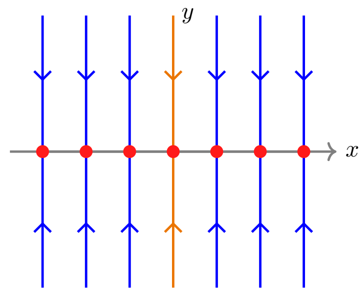
Problem 5. *A system $\mathbf{x}' = A\mathbf{x}$ has solution $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.*

Sketch a phase portrait.

Solution: Since one eigenvalue $\lambda = 0$, this is the degenerate case.

Each of the solutions $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an equilibrium, i.e., a constant.

A mixed mode, e.g., $\mathbf{x}(t) = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a straight line that goes asymptotically to $2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as $t \to \infty$. Other mixed modes are similar.



This is called degenerate because the critical point at (0,0) is not isolated.

Dynamical stability of each equilibrium point: edge case

The system is structurally unstable.

Problem 6. *A system* $\mathbf{x}' = A\mathbf{x}$ *has solution* $\mathbf{x}(t) = c_1 e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{2t}\left(t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\right)$.

Sketch a phase portrait.

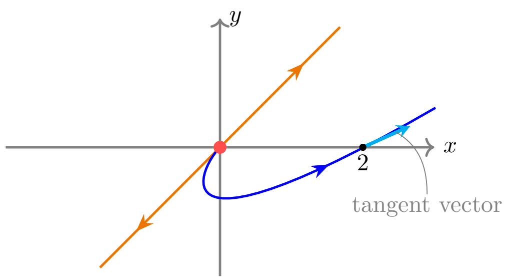
Solution: Repeated positive roots. From the solution, there is only one mode, i.e., $c_1 e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$. So this is a defective nodal source.

The one mode plays both asymptotic roles.

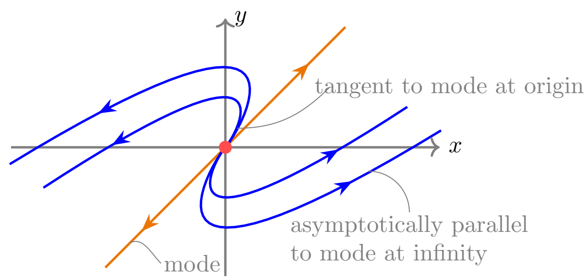
As $t \to \infty$, trajectories are asymptotically parallel to the mode.

As $t \to -\infty$, trajectories are asymptotically tangent to the mode as they approach 0.

Sense of turning: Take $\mathbf{x}(t) = e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + e^{2t}\left(t\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\right)$. Then $\mathbf{x}(0) = \begin{bmatrix}2\\0\end{bmatrix}$, $\mathbf{x}'(0) = \begin{bmatrix}5\\1\end{bmatrix}$. Since this points up from the axis, the trajectory must turn counterclockwise.



Tangent vector gives the sense of turning



Phase portrait

The equilibrium at (0,0) is dynamically unstable. The system is structurally unstable.