Day 57, R 5/2/2024

Topic 28: Nonlinear systems
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1 Agenda

- Phase portraits for nonlinear systems: critical point analysis
- Reason for the linearization formula (probably next time)

2 Introduction to nonlinear autonomous systems

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

Autonomous because no t in $f(x, y)$, $g(x, y)$, i.e., x, y control their own rate of change.

Example 1.

$$\begin{aligned} x' &= 14x - \frac{1}{2}x^2 - xy \\ y' &= 16y - \frac{1}{2}y^2 - xy \end{aligned}$$

(This could model the populations of two interacting species.)

2.1 Critical point analysis: linearization at a critical point

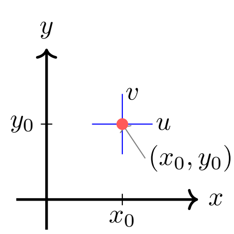
Critical points are points (x_0, y_0) where

$$\left.\begin{aligned} x' &= f(x_0, y_0) = 0 \\ y' &= g(x_0, y_0) = 0 \end{aligned}\right\} \longleftrightarrow \text{ equilibrium solutions } x(t) = x_0,\ y(t) = y_0.$$

Jacobian: $\quad J(x_0, y_0) = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}$

Linearization at a critical point: Near a critical point (x_0, y_0), the system is approximated by the linear system

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = J(x_0, y_0) \begin{bmatrix} u \\ v \end{bmatrix}$$



Example 2. (Critical point analysis) Consider the nonlinear system

$$x' = 14x - \frac{1}{2}x^2 - xy = f(x,y)$$
$$y' = 16y - \frac{1}{2}y^2 - xy = g(x,y).$$

(a) If x and y are two populations, describe their interactions.

(b) Find the critical points of the system.

(c) Linearize at each critical point. Consider structural stability and what this says about the nonlinear system

(d) On the phase plane (xy-plane) plot each of the critical points and some nearby trajectories.

(e) Tie (d) together into a phase portrait of the nonlinear system.

(f) Interpret the portrait, e.g., if x, y are populations, describe how they evolve.

Solution: **(a)** The interaction term $-xy$ is negative in both equations. So interaction lowers the growth rate of each. The species are competing.

Without the presence of the other, each species has logistic growth.

(b) Critical points:

$$\begin{aligned} x' &= 14x - \frac{1}{2}x^2 - xy &= x\left(14x - \frac{1}{2}x - y\right) &= 0 \\ y' &= 16y - \frac{1}{2}y^2 - xy &= y\left(16y - \frac{1}{2}y - x\right) &= 0 \end{aligned}$$

In each of the equations one of the factors must be 0.

Cases:

$$\left.\begin{array}{ll} x = 0,\ y = 0 & \rightsquigarrow \text{critical point } (0,0) \\ x = 0,\ 16 - \frac{1}{2}y - x = 0 & \rightsquigarrow \text{critical point } (0,32) \\ 14 - \frac{1}{2}x - y = 0,\ y = 0 & \rightsquigarrow \text{critical point } (28,0) \\ \underbrace{14 - \frac{1}{2}x - y = 0,\ 16 - \frac{1}{2}y - x = 0}_{\text{solve simultaneous equations}} & \rightsquigarrow \text{critical point } (12,8) \end{array}\right\} \text{ all the critical points.}$$

(c) To linearize, we find the Jacobian at each critical point.

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 14 - x - y & -x \\ -y & 16 - y - x \end{bmatrix}.$$

<u>At $(0,0)$</u>: $J(0,0) = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix} \longrightarrow \lambda = 14,\ 16.$

$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$ is a linearized nodal source.

Structurally stable $\longrightarrow$ nonlinear nodal source.

<u>At $(0,32)$</u>: $J(0,32) = \begin{bmatrix} -18 & 0 \\ -32 & 16 \end{bmatrix} \longrightarrow \lambda = -18,\ -16.$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -18 & 0 \\ -32 & 16 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{is a linearized nodal sink.}$$

Structurally stable $\longrightarrow$ nonlinear nodal sink.

<u>At $(28, 0)$</u>: $\quad J(28,0) = \begin{bmatrix} -14 & -28 \\ 0 & -12 \end{bmatrix} \longrightarrow \lambda = -14, -12.$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -14 & -28 \\ 0 & -12 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{is a linearized nodal sink.}$$

Structurally stable $\longrightarrow$ nonlinear nodal sink.

<u>At $(12, 8)$</u>: $\quad J(12,8) = \begin{bmatrix} -6 & -12 \\ -8 & -4 \end{bmatrix} \longrightarrow \lambda = -5 \pm \sqrt{97} \approx -15,\ 5.$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -6 & -12 \\ -8 & -4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{is a linearized saddle.}$$

Structurally stable $\longrightarrow$ nonlinear saddle.

(d) , **(e)** For sinks and sources, there is no need to find eigenvectors since they are only valid near the critical point and the key fact is that all trajectories go towards (sinks) or away (sources) from the critical point.

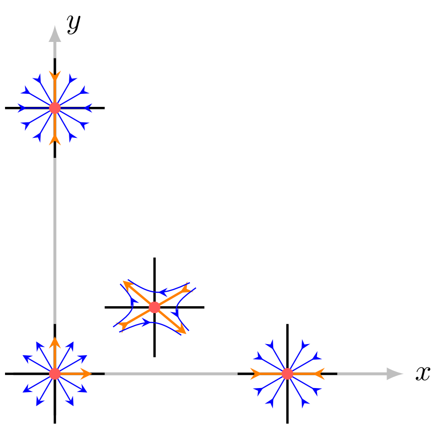
For saddles, finding eigenvectors helps to orient the sketch.

<u>At $(12, 8)$</u>: $\quad J(12,8) = \begin{bmatrix} -6 & -12 \\ -8 & -4 \end{bmatrix}$ has $\lambda = -5 \pm \sqrt{97}$.

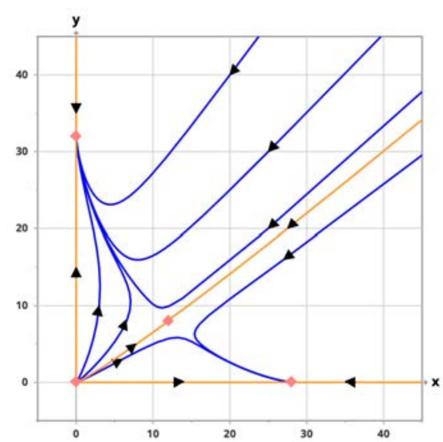
Eigenvectors:

$\lambda = -5+\sqrt{97}$: $\quad A - \lambda I = \begin{bmatrix} -1-\sqrt{97} & -12 \\ -8 & 1-\sqrt{97} \end{bmatrix} \to$ basic eigenvector $= \begin{bmatrix} 1-\sqrt{97} \\ 8 \end{bmatrix} \approx \begin{bmatrix} -9 \\ 8 \end{bmatrix}$.

$\lambda = -5-\sqrt{97}$: $\quad A - \lambda I = \begin{bmatrix} -1+\sqrt{97} & -12 \\ -8 & 1+\sqrt{97} \end{bmatrix} \to$ basic eigenvector $= \begin{bmatrix} 1+\sqrt{97} \\ 8 \end{bmatrix} \approx \begin{bmatrix} 11 \\ 8 \end{bmatrix}$.



(d) Critical points and nearby trajecories



(e) Phase portrait tying (d) together

3 Reason for linearization formula

This is the usual linear (or tangent plane) approximation from 18.02.

For a function $f(x, y)$, near a point (x_0, y_0) we have

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y, \quad \text{where } \Delta x = x - x_0,\ \Delta y = y - y_0.$$

Say (x_0, y_0) is a critical point and let $u = x - x_0$, $v = y - y_0$.

Keep in mind:

1. $f(x_0, y_0) = 0$, $g(x_0, y_0) = 0$.
2. Since x_0 is constant, $u' = x'$. Likewise, $v' = y'$.

Using the approximation formula

$$\begin{aligned} u' = x' = f(x,y) &\approx \overbrace{f(x_0,y_0)}^{0} + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y = f_x(x_0,y_0)\,u + f_y(x_0,y_0)\,v \\ v' = y' = g(x,y) &\approx \underbrace{g(x_0,y_0)}_{0} + g_x(x_0,y_0)\Delta x + g_y(x_0,y_0)\Delta y = g_x(x_0,y_0)\,u + g_y(x_0,y_0)\,v \end{aligned}$$

Looking at the first and last terms, in matrix form, we have

$$\begin{bmatrix} u' \\ v' \end{bmatrix} \approx \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = J(x_0,y_0) \begin{bmatrix} u \\ v \end{bmatrix}.$$

This is the linear approximation formula near a critical point!