Solutions Day 57, R 5/2/2024

Topic 28: Nonlinear Systems

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Problem 1. *Consider the system*

$$x' = 3x - x^2 - xy$$
$$y' = y - y^2 + xy$$

(a) *Find the critical points*

(b) *Linearize at each critical point. Say what this implies about the nonlinear system.*

(c) *On the xy-plane, show each of the critical points and the linear approximation near it.*

(d) *Tie Part (c) together into a phase-portrait of the nonlinear system.*

(e) *Suppose x, y represent populations of two species, tell a story about their interaction.*

Solution: **(a)** Critical points $\begin{array}{lll} x' & = x(3-x-y) & = 0 \\ y' & = y(1-y+x) & = 0 \end{array}$.

Cases:

$$\left.\begin{array}{ll} x=0, y=0 & \rightsquigarrow (0,0) \\ x=0, 1-y+x=0 & \rightsquigarrow (0,1) \\ 3-x-y=0, y=0 & \rightsquigarrow (3,0) \\ 3-x-y=0, 1-y+x=0 & \rightsquigarrow (1,2) \end{array}\right\} \text{ all the critical points.}$$

(b) Jacobian: $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 3-2x-y & -x \\ y & 1-2y+x \end{bmatrix}$.

<u>At $(0,0)$</u>: $J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. So, $\lambda = 3,\ 1$.

Linearized nodal source. Structurally stable implies nonlinear nodal source.

<u>At $(0,1)$</u>: $J(0,1) = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$. So, $\lambda = 2,\ -1$.

Linearized saddle. Structurally stable implies nonlinear saddle.

<u>At $(3,0)$</u>: $J(3,0) = \begin{bmatrix} -3 & -3 \\ 0 & 4 \end{bmatrix}$. So, $\lambda = -3,\ 4$.

Linearized saddle. Structurally stable implies nonlinear saddle.

<u>At $(1,2)$</u>: $J(1,2) = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$. So, $\lambda = \dfrac{-3 \pm \sqrt{7}\,i}{2}$.

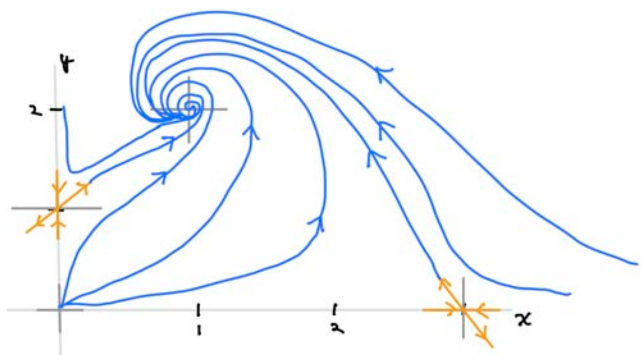
Linearized spiral sink. Structurally stable implies nonlinear spiral sink.

(c) , **(d)** Eigenvectors for saddles.

At $(0,1)$: $\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$ has eigenpairs $\begin{cases} \lambda = 2 & \lambda = -1 \\ \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} & \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$

At $(3,0)$: $\begin{bmatrix} -3 & -3 \\ 0 & 4 \end{bmatrix}$ has eigenpairs $\begin{cases} \lambda = -3 & \lambda = 4 \\ \mathbf{v} = \begin{bmatrix} 3 \\ -7 \end{bmatrix} & \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$

Here is a hand-drawn phase portrait:



(e) The $-xy$ term in x' tells us that the presence of y decreases the growthrate of x. Likewise, the $+xy$ term in y' says that the presence of x increases the growthrate of y. This is a predator-prey relationship, with x = prey and y = predator.

All trajectories go to the spiral sink at $(1,2)$. That is, for all initial values, the populations will stabilize at this point.

Problem 2. *Consider the system*

$$x' = 2x - 3xy$$
$$y' = -y + 2xy$$

Answer the same questions as in Problem 1.

Note: One of the critical points is not structurally stable. So you will have to entertain several possibilities for its type.

Solution: (a) Critical points $\begin{aligned} x' &= x(2-3y) &= 0 \\ y' &= y(-1+2x) &= 0 \end{aligned}$.

Cases:

$$\begin{array}{ll} x = 0, y = 0 & \rightsquigarrow (0,0) \\ x = 0, -1+2x = 0 & \rightsquigarrow \text{ no solutions} \\ 2-3y = 0, y = 0 & \rightsquigarrow \text{ no solutions} \\ 2-3y = 0, -1+2x = 0 & \rightsquigarrow (1/2,\ 2/3) \end{array}$$

So there are critical points at (0,0), (1/2, 2/3).

(b) Jacobian $J(x,y) = \begin{bmatrix} 2-3y & -3x \\ 2y & -1+2x \end{bmatrix}$

<u>At $(0,0)$:</u> $J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. So, $\lambda = 2, -1$.

Linearized saddle. Structurally stable implies nonlinear saddle.

<u>At $\left(\frac{1}{2}, \frac{2}{3}\right)$:</u> $J\left(\frac{1}{2}, \frac{2}{3}\right) = \begin{bmatrix} 0 & -3/2 \\ 4/3 & 0 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + 2 = 0 \quad \Rightarrow \lambda = \pm 2i.$

Linearized center. Not structurally stable, so could be a nonlinear center, spiral sink or spiral source.

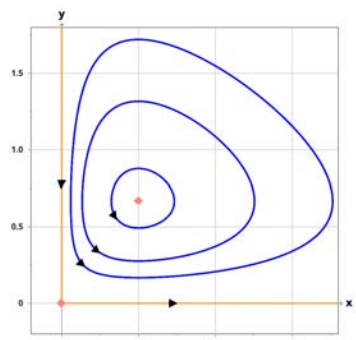
(c) , **(d)** Eigenvectors for saddle:

$$J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{diagonal, so eigenpairs are } \left\{ \begin{matrix} 2 & -1 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{matrix} \right.$$

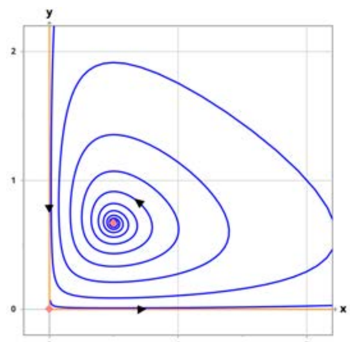
Sense of center/spiral: $J\left(\frac{1}{2}, \frac{2}{3}\right) = \begin{bmatrix} 0 & -3/2 \\ \textcircled{4/3} & 0 \end{bmatrix}.$

The 4/3 in the lower left is positive, so the center or spiral turns counterclockwise.

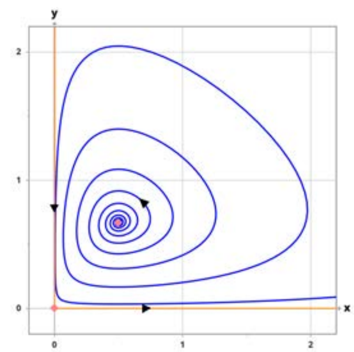
For the graphs, we need to show all 3 possibilities.



Center at (1/2, 2/3)



Spiral sink at (1/2, 2/3)



Spiral source at (1/2, 2/3)

(e) As in Problem 1, this is a predator-prey relationship, with x = prey and y = predator. Unlike Problem 1, in the absence of y, x grows exponentially. And, in the absence of x, y decays exponentially.

Because the critical point at (1/2, 2/3) could be a center, spiral source or spiral sink, we can't say exactly how the populations evolve.

Note: With more work (which we won't do in ES.1803), it is possible to show that (1/2, 2/3) is a nonlinear center. So, the populations will circle around the critical point.