Day 59, M 5/6/2024

Topic 29: Structural stability
Jeremy Orloff

1 Agenda

- Finish nonlinear phase portraits from previous class
- Structural stability

2 Structural stability

The Topic 29 notes show some tricks for analyzing critical points with non-structurally stable linearizations. This is interesting and important, but tangential to the class. Read it for enrichment.

Example 1. (Linearized centers can really be spirals or centers)

Consider the following systems, all with a linearized center at $(0,0)$,

System 1: $\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$

This system is already linear: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The critical point at $(0,0)$ is a center.

System 2: $\begin{aligned} x' &= y \\ y' &= -x - y^3 \end{aligned}$

$(0,0)$ is a critical point. $J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longrightarrow$ linearized center.

With more work we can show that $(0,0)$ is a nonlinear spiral sink. The Topic 29 notes show how to prove this using Lyuponov's second method. This mimics the use of potential functions in physics. The argument is outlined below.

System 3: $\begin{aligned} x' &= y \\ y' &= -x + y^3 \end{aligned}$

$(0,0)$ is a critical point. $J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longrightarrow$ linearized center.

With more work we can show that $(0,0)$ is a nonlinear spiral source. (See the Topic 29 notes or the outline below.)

The following applet illustrates non-structurally stable critical points.

Applet: https://web.mit.edu/jorloff/www/OCW-ES1803/vectorFields.html

Choose the system $\left[\begin{aligned} x' &= y \\ y' &= -x + a \cdot \mathrm{sign}(y)\, y^2 \end{aligned}\right]$, and change the parameter a between positive, negative and 0. (For numerical reasons, the applet uses $a \cdot \mathrm{sign}(y)\, y^2$ instead of ay^3, but the idea is the same as the example above.)

2.1 Outline of the argument for System 2

The argument is known as Lyapunov's second method. It borrows the notion of a potential function from physics.

Let $V = x^2 + y^2$. So the level curves of V are circles. Think of V as potential energy for our dynamical system.

If $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ solves System 2, i.e. $\begin{aligned} x' &= y \\ y' &= -x - y^3 \end{aligned}$, then

$$\frac{d}{dt}V(x(t), y(t)) = 2xx' + 2yy' = 2xy + 2y(-x - y^3) = -2y^4 \leq 0.$$

So $V(x(t), y(t))$ is decreasing. Since the level curves of V are circles, the trajectory must keep going to smaller and smaller circles, i.e., must go asymptotically to $(0, 0)$. Thus it must be a spiral and not a closed loop.