Solutions Day 6, M 2/12/2024
Topic 4: Complex numbers (day 1)
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Problem 1. *Let $z_1 = 2 + 5i, \quad z_2 = 1 + 3i$*

(a) *Compute $z_1 + z_2$, $z_1 \cdot z_2$, $z_1 \cdot \overline{z_1}$, $|z_1|$, $\text{Arg}(z_1)$.*

Solution: $z_1 + z_2 = 3 + 8i$.

$z_1 \cdot z_2 = (2 + 5i)(1 + 3i) = 2 - 15 + 6i + 5i = -13 + 11i$.

$z_1 \cdot \overline{z_1} = (2 + 5i)(2 - 5i) = 4 + 25 = 29$.

$|z_1| = \sqrt{2^2 + 5^2} = \sqrt{29}$.

$\text{Arg}(z_1) = \tan^{-1}(5/2)$ in Q1.

(b) *Find $\text{Re}(z_1)$, $\text{Im}(z_1)$.*

Solution: $\text{Re}(z_1) = 2, \quad \text{Im}(z_1) = 5$. (Imaginary part is a real number!)

(c) *Let $z = x + iy$. Compute $z \cdot \overline{z}$.*

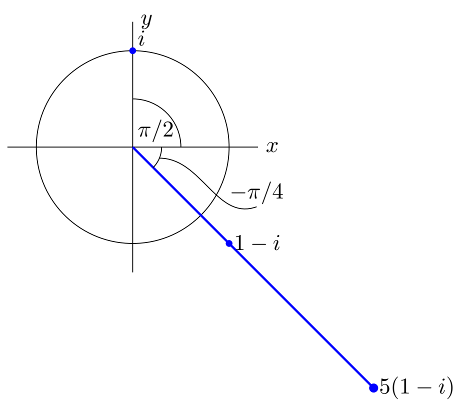
Solution: $z \cdot \overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$.

Problem 2. *Find r and $\theta = \text{Arg}(z)$ for $z = i, \quad z = 1 - i, \quad z = 5(1 - i)$.*

Solution: $z = i$: $\quad |i| = 1, \quad \theta = \text{Arg}(i) = \pi/2 + 2n\pi$, n an integer.

$z = 1 - i$: $\quad |1 - i| = \sqrt{2}, \quad \theta = \text{Arg}(1 - i) = -\pi/4 + 2n\pi$, n an integer.

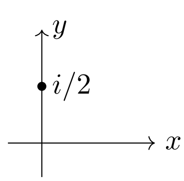
$z = 5(1 - i)$: $\quad |5(1 - i)| = 5\sqrt{2}, \quad \theta = \text{Arg}(5(1 - i)) = -\pi/4 + 2n\pi$, n an integer.



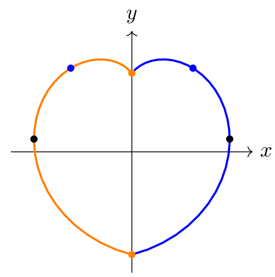
Problem 3.

(a) *Write $\frac{i}{2}$ in polar form.*

Solution: Looking at the picture, we see $\left|\frac{i}{2}\right| = \frac{1}{2}$, $\text{Arg}\left(\frac{i}{2}\right) = \frac{\pi}{2}$. So, $\boxed{\frac{i}{2} = \frac{1}{2}e^{i\pi/2}}$.

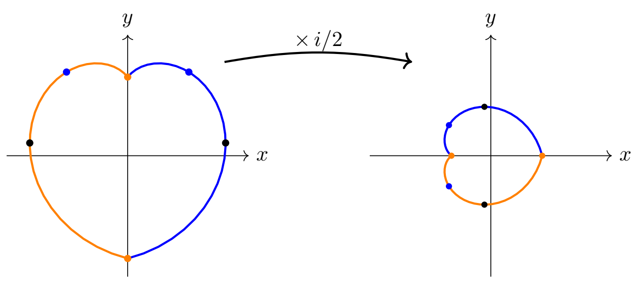


(b) *Consider the diagram*



Multiply the diagram by $i/2$, i.e., sketch the resulting image.

Solution: If $z = re^{i\theta}$ then $\frac{i}{2} \cdot z = \frac{1}{2}e^{i\pi/2} \cdot re^{i\theta} = \frac{r}{2}e^{i(\theta+\pi/2)}$. So multiplication by $i/2$ scales z by $1/2$ and rotates it by $\pi/2$ counterclockwise. The sketch is below.



Problem 4. *Show $\overline{e^{i\theta}} = e^{-i\theta}$.*

Solution: $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$ ■

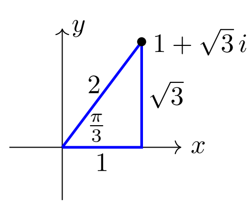
Problem 5. *Compute $(1+\sqrt{3}\,i)^{10}$. (Use polar form.)*

Solution: We write $1+\sqrt{3}i$ in polar form.

Magnitude: $|1+\sqrt{3}\,i| = 2$.

Argument: We recognize the 30°, 60°, 90° triangle. So, $\text{Arg}(1+\sqrt{3}\,i) = \pi/3$.

Thus, $1+\sqrt{3}\,i = 2e^{i\pi/3}$.



Thus, $(1+\sqrt{3}\,i)^{10} = \left(2e^{i\pi/3}\right)^{10} = 2^{10}e^{i10\pi/3} = 2^{10}e^{i4\pi/3} = 2^{10} \cdot \left(\frac{-1-\sqrt{3}\,i}{2}\right)$.

(The step leading to $2^{10}e^{i4\pi/3}$ is because $10\pi/3 = 4\pi/3 + 2\pi$.)

Problem 6. *Compute $I = \int e^x \cos(5x)\,dx$.*

Solution: Complexify: $I_c = \int e^x e^{5xi}\,dx = \int e^{x(1+5i)}\,dx.$

Relation: $I = \mathrm{Re}(I_c)$.

Computing: $I_c = \dfrac{e^{x(1+5i)}}{1+5i}$.

We need to find the real part of I_c. In 18.03 we use polar form, so first we need to find the polar form of $1+5i$.

$$|1+5i| = \sqrt{26}; \quad \boxed{\phi = \mathrm{Arg}(1+5i) = \tan^{-1}(5) \text{ in Q1}}.$$

So, $1+5i = \sqrt{26}e^{i\phi}$. This gives, $I_c = \dfrac{e^{(1+5i)x}}{1+5i} = \dfrac{e^x e^{5xi}}{\sqrt{26}e^{i\phi}} = \dfrac{e^x}{\sqrt{26}}\,e^{i(5x-\phi)}$.

Expanding this: $I_c = \dfrac{e^x}{\sqrt{26}}\left(\cos(5x-\phi) + i\sin(5x-\phi)\right)$.

Thus, $\boxed{I = \mathrm{Re}(I_c) = \dfrac{e^x}{\sqrt{26}}\cos(5x-\phi)}$.

Note: As we get comfortable with this, we will skip writing down so many of the steps.

Problem 7.
(a) *Find the fifth roots of 1. Draw a picture.*

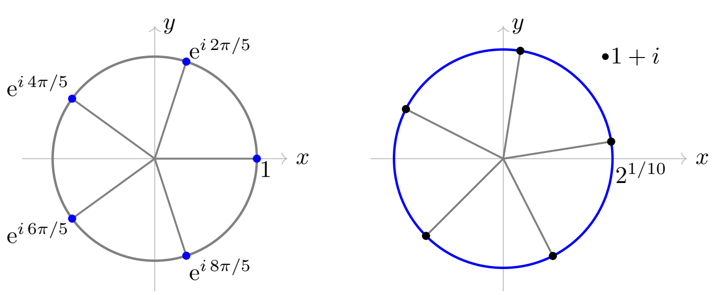
Solution: Write 1 in polar form with all possible θ:

$$1 = e^{i2\pi n},\ n \text{ any integer.}$$

We want z so that $z^5 = 1 = e^{i\cdot 2\pi n}$. So,

$$z = e^{i\cdot 2\pi n/5} = \underbrace{\overbrace{e^0}^{1},\ e^{i\cdot 2\pi/5},\ e^{i\cdot 4\pi/5},\ e^{i\cdot 6\pi/5},\ e^{i\cdot 8\pi/5}}_{5 \text{ fifth roots}},\ \overbrace{e^{i10\pi/5}}^{\text{same as } e^0=1},\ \ldots$$

On the plot, the roots are evenly spaced every $2\pi/5 = 72^\circ$ around the unit circle.



Left: picture for Part (a) right: picture for Part (b)

(b) *Find the fifth roots of $1+i$. Draw a picture.*

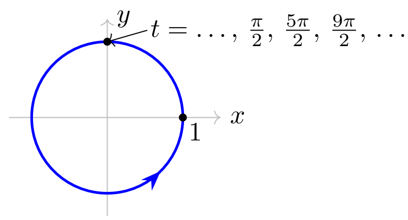
Solution: The idea is the same idea as Part (a). We have $1+i = \sqrt{2}e^{i\pi/4+2n\pi}$. So, $z^5 = 1+i \quad \Rightarrow z = 2^{1/10}e^{i(\pi/20+2n\pi/5)}$. The 5 roots are

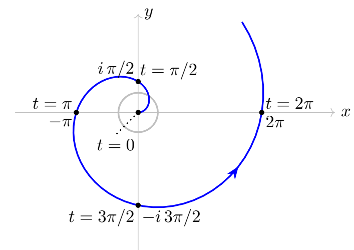
$$2^{1/10}e^{i\pi/20},\ 2^{1/10}e^{i9\pi/20},\ 2^{1/10}e^{i17\pi/20},\ 2^{1/10}e^{i25\pi/20},\ 2^{1/10}e^{i33\pi/20}.$$

Graphically the five roots are evenly spaced around the circle of radius $2^{1/10}$.

Problem 8.
(a) *Draw the trajectory of $z = e^{it}$*

Solution: $z = e^{it} = \cos t + i \sin t$. This always has unit length. So the trajectory goes round and round the unit circle.





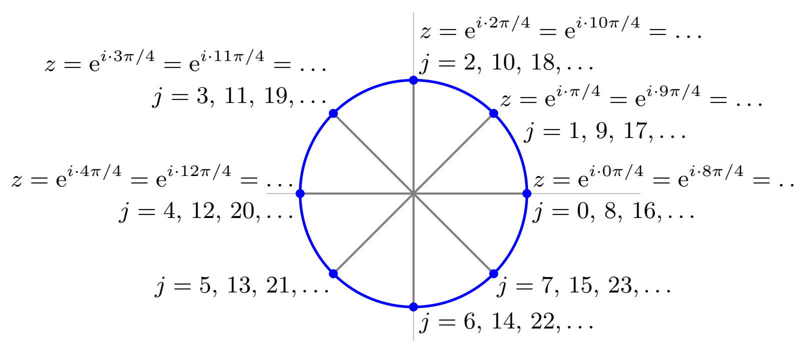
Left: plot for Part (a); right: plot for Part (b)

(b) *Draw the trajectory of $z = te^{it}$*

Solution: This is similar to Part (a), except the factor of t causes the magnitude of z to grow. The trajectory spirals out from the origin.

(c) *Plot the points $e^{ij\pi/4}$, for $j = 0, 1, 2, 3$ ...*

Solution: Note: $e^{i8\pi/4} = e^{i2\pi} = 1$. These are the 8th roots of 1, i.e., $z^8 = 1$.



Problem 9.
(a) *Write $\sin t$ and $\cos t$ in terms of e^{it} and e^{-it}.*

Solution: Expand the exponentials: $\begin{cases} e^{it} &= \cos t + i \sin t \\ e^{-it} &= \cos t - i \sin t \end{cases}$.

Adding and subtracting: $\begin{cases} e^{it} + e^{-it} &= 2\cos t \\ e^{it} - e^{-it} &= 2i \sin t \end{cases}$. So, $\boxed{\begin{aligned} \cos t &= \frac{e^{it} + e^{-it}}{2} \\ \sin t &= \frac{e^{it} - e^{-it}}{2i} \end{aligned}}$.

(b) *Find all <u>real-valued</u> functions of the form $f(t) = c_1 e^{it} + c_2 e^{-it}$, where c_1, c_2 are complex constants.*

Solution: From Part (a), we see that

$$\frac{1}{2}e^{it} + \frac{1}{2}e^{-it} = \cos t \quad \text{and} \quad \frac{1}{2i}e^{it} - \frac{1}{2i}e^{-it} = \sin t$$

are real-valued linear combinations of e^{it} and e^{-it}. Thus, for a, b real numbers,

$$\left(\frac{a}{2} + \frac{b}{2i}\right) e^{it} + \left(\frac{a}{2} - \frac{b}{2i}\right) e^{-it} = a\cos t + b\sin t$$

are the real-valued linear combinations of e^{it} and e^{-it}.

Problem 10. *Find all the roots of* $x^4 + x^2 = 0$.

Solution: This is fourth-order, so there are 4 roots. Factoring:

$$x^2(x^2+1) = 0 \qquad \Rightarrow x = 0,\ 0,\ i,\ -i \text{ are the 4 roots.}$$