Day 60, T 5/7/2024

Topic 30: Population models

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1 Agenda

- Volterra's Principle
- Armand and Babette (in problems)

2 Volterra predator-prey model

x, y are two populations, $x =$ prey, $y =$ predator.

$$\begin{aligned} x' &= ax - pxy \\ y' &= -by + qxy, \end{aligned} \qquad a,\ b,\ p,\ q \text{ are positive constants}$$

Let's find the critical points and make a phase portrait.

Critical points:

$$\begin{aligned} x' &= ax - pxy &&= x(a - py) &&= 0 \\ y' &= -by + qxy &&= y(-b + qy) &&= 0 \end{aligned}$$

There are two critical points: $(0,0)$, $(b/q,\ a/p)$.

Linearize at each critical point.

Jacobian $J(x,y) = \begin{bmatrix} a - py & -px \\ qy & -b + qx \end{bmatrix}$

$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & -b \end{bmatrix}$.

Eigenvalues: a $\quad -b$

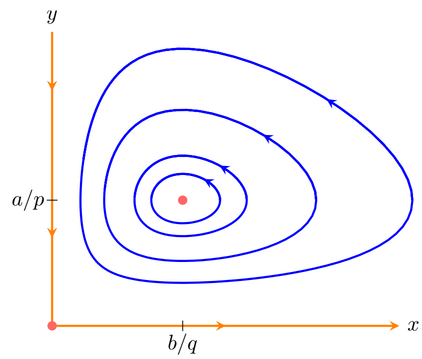
Eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Linearized saddle. Structurally stable $\longrightarrow$ nonlinear saddle.

$J(b/q,\ a/p) = \begin{bmatrix} 0 & -bp/q \\ aq/p & 0 \end{bmatrix}$. Characteristic equation: $\lambda^2 + ab = 0 \longrightarrow \lambda = \pm\sqrt{ab}\, i$

Linearized center. Not structurally stable, so could be a center, spiral sink spiral source.
With more work (see the Topic 30 notes), we can show it is a nonlinear center.

Since the (2,1) entry in the Jacobian is $\dfrac{aq}{p} > 0$, trajectories turn counterclockwise.



3 Volterra Principle

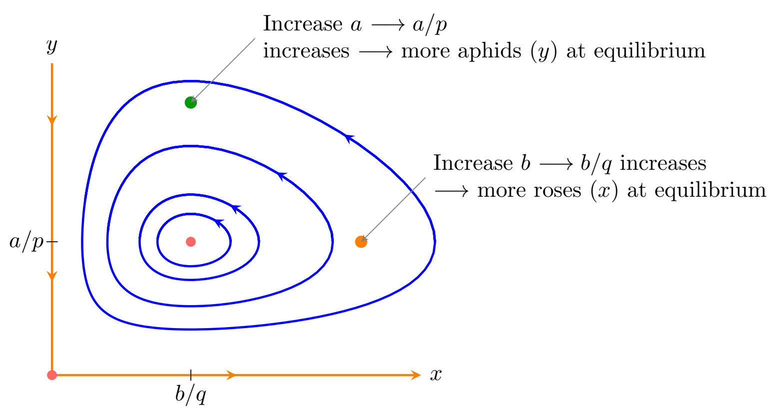
Say x = roses (prey), y = aphids (predator).

Aphids eat roses. If I want more roses, here are two possible strategies:

Strategy 1: Feed the roses, i.e., increase the growth rate a.

Strategy 2: Poison the aphids, i.e., increase the decay rate b.

Which strategy is better? Here is a phase portrait showing how the critical point moves under the two strategies.



Increasing a causes a/p to increase. So the new critical point (green dot on the above phase portrait) has the same x (roses) coordinate and a bigger y (aphids) coordinate. Feeding the roses led to more aphids to eat the faster growing roses. The rose population was unchanged.

Increasing b causes b/q to increase. So the new critical point (orange dot on the above phase portrait) has the same y (aphids) coordinate and a bigger x (roses) coordinate. Poisoning the aphids led to more roses. The aphid population was unchanged.

So Strategy 2 is better at increasing rose production.

Volterra: During WWI, the Italian fishing boats couldn't go out because of the German u-boats. After the war, they were surprised to find the same amount of fish, but more sharks in the sea. This led Volterra to develop his model and principle.