Solutions Day 60, T 5/7/2024
Topic 30: Population models
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Problem 1. *Fancier predator-prey (we've looked at this system once before):*

$$x' = 3x - x^2 - xy$$
$$y' = y - y^2 + xy$$

(a) *Which is predator? prey?*

Solution: x = prey (the $-xy$ term); y = predator (the $+xy$ term).

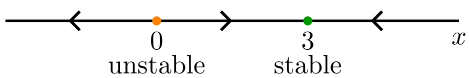
(b) *If there is no y ($y = 0$), what does the model say about x?*

What is the model called in this case?

Solution: If $y = 0$, the model says $x' = 3x - x^2 = (3 - x)x$.

This is a logistic population model that we looked at in Topic 12.

We can analyze it by drawing its phase line, i.e., a one dimensional phase portrait.



(c) *Here is a table of critical points*

Critical points	(x_0, y_0) :	$(0,0)$	$(0,1)$	$(3,0)$	$(1,2)$
Jacobian	$J(x_0, y_0)$:	$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} -3 & -3 \\ 0 & 4 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$
Eigenvalues	λ :	$3, 1$	$2, -1$	$-3, 4$	$\dfrac{-3 \pm \sqrt{7}\, i}{2}$
Eigenvectors (if needed) :			$\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \end{bmatrix}$	

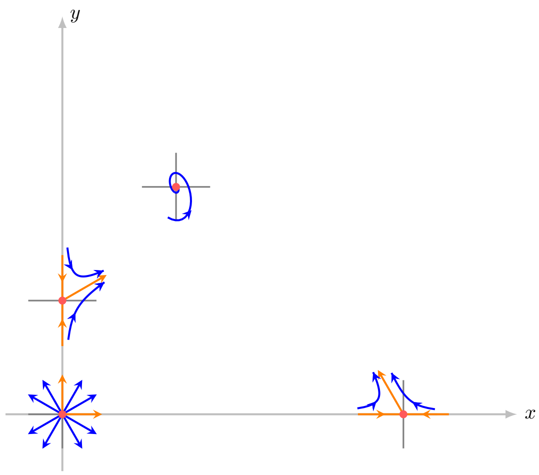
Draw a phase portrait.

Solution: Using the table we have the following types of critical points

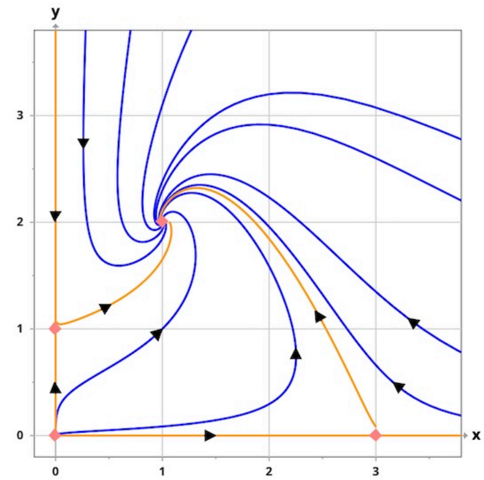
Critical point at	(0,0)	is a linearized	nodal source.
Critical point at	(0,1)	is a linearized	saddle.
Critical point at	(3,0)	is a linearized	saddle.
Critical point at	(1,2)	is a linearized	spiral sink.

Since all of these are structurally stable, they also describe the nonlinear system at the critical points.

The spiral sink turns counterclockwise (implied by the 2 in the bottom left entry of the Jacobian).



4 linearizations in phase plane



Phase portrait

(d) *Tell a story.*

Solution: No matter what the initial conditions, as long as $x(0) > 0$ and $y(0) > 0$, the populations go asymptotically to $(1, 2)$.

Problem 2. (Armand and Babette go nonlinear (Pset 9))
We won't copy down their story. The system is

$$x' = x - 2y + \frac{1}{4}x^2$$
$$y' = 5x - y - y^2.$$

(a) *Find the critical points. (Hint: you'll end up with a quartic polynomial. One root is 0, another is a positive integer ≤ 5.)*

Solution: We have to solve $\begin{aligned} x' &= x - 2y + \tfrac{1}{4}x^2 &= 0 \\ y' &= 5x - y - y^2 &= 0 \end{aligned}$.

Using the second equation, we get $x = \dfrac{y + y^2}{5}$. Substituting this into the first equation gives

$$x - 2y + \frac{1}{4}x^2 = \frac{y + y^2}{5} - 2y + \frac{(y + y^2)^2}{100} = 0 \quad \Rightarrow y^4 + 2y^3 + 21y^2 - 180y = 0.$$

Factoring: $y(y^3 + 2y^2 + 21y - 180) = 0$. So, $y = 0$ and (by testing small positive integers) $y = 4$ are roots.

Factoring again: $y(y - 4)(y^2 + 6y + 45) = 0$. Solving the quadratic, we find all the roots are

$$y = 0,\ 4,\ -3 \pm 6i.$$

Only real roots give critical points. So, using $x = (y + y^2)/5$, we have

$$y = 0 \quad \Rightarrow x = 0; \qquad y = 4 \quad \Rightarrow x = 4, \qquad \text{i.e., the critical points are } (0, 0),\ (4, 4).$$

(b) *Linearize at each critical point and sketch the phase portrait of the nonlinear system.*

Solution: Jacobian $J(x,y) = \begin{bmatrix} 1+x/2 & -2 \\ 6 & -2-2y \end{bmatrix}$.

<u>At $(0,0)$</u>: $\quad J(0,0) = \begin{bmatrix} 1 & -2 \\ 6 & -2 \end{bmatrix}$.

Characteristic equation: $\quad \lambda^2 + \lambda + 10 = 0 \quad \Rightarrow \lambda = \frac{-1 \pm \sqrt{39}\, i}{2}$.

This is a linearized spiral sink. Since it's structurally stable, it is also a nonlinear spiral sink. The 6 in the bottom left of the Jacobian implies it turns counterclockwise.

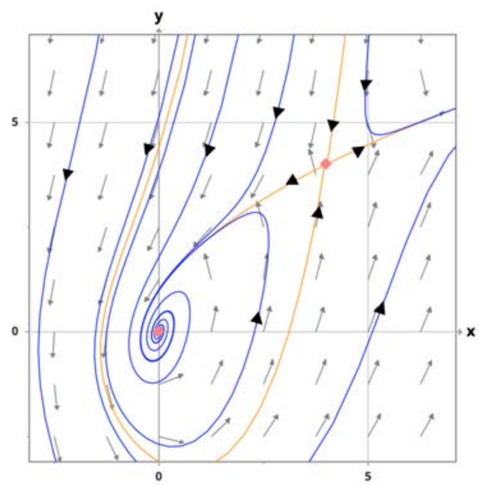
<u>At $(4,4)$</u>: $\quad J(4,4) = \begin{bmatrix} 3 & -2 \\ 6 & -10 \end{bmatrix}$.

Characteristic equation: $\quad \lambda^2 + 7\lambda - 18 = 0 \quad \Rightarrow \lambda = -9\, 2$.

This is a linearized saddle. Since it's structurally stable, it is also a nonlinear saddle.

We compute the eigenvectors to help with the phase portrait:

$$\begin{aligned} \lambda &= -9 \quad 2 \\ \mathbf{v} &= \begin{bmatrix} 1 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned}$$



Phase portrait

(c) *Interpret the results in terms of their relationship.*

Solution: Depending on where their trajectory starts, their mutual feelings could get sucked into the vortex which is the spiral sink at the origin. Sadly their mutual attraction would dwindle to nothing.

Or they could be rocketed from the saddle off to mutually infinite love. Some of these trajectories never go negative. Others spend a short time outside the first quadrant, indicating a rough patch in their relationship before they get their act together and begin to truly love each other.