Day 9, R 2/15/2024

Topic 5: Homogeneous, linear, constant coefficient DEs (day 1 of 2)

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1 Agenda

- Super important in 1803 and beyond
- All about e^{rt}
- Algebraic methods (method of optimism)
- Superposition principle (linearity)
- Complex roots
- Repeated roots
- Damped harmonic oscillators (probably tomorrow)
- Existence and uniqueness theorem (probably tomorrow)
- Pole diagrams (probably tomorrow)

2 Second-order, homogeneous, constant coefficent DEs

Example 1. Here are two linear, second-order, homogeneous, constant coefficent DEs:

$$x'' + 8x' + 7x = 0$$
$$3x'' + 5x' + 9x = 0$$

Second-order: Highest derivative is order 2.

Linear: No powers or products with the x's.

Constant coefficent: Coefficients of x'', x', x are constants.

Homogeneous: With all the x's on the left side, the right side is 0.

Example 2. (Solving a second-order, linear, constant coefficient, homogeneous DE)

Solve $x'' + 8x' + 7x = 0$.

Solution: (Model solution, reasons below)

Characteristic equation: $r^2 + 8r + 7 = 0$

Roots: $(r + 1)(r + 7) = 0 \quad \longrightarrow \quad r = -1, -7$

Basic (modal) solutions: $x_1(t) = e^{-t}, \quad x_2(t) = e^{-7t}$.

General solution by superposition: $x(t) = c_1x_1(t) + c_2x_2(t) = c_1e^{-t} + c_2e^{-7t}$ (c_1, c_2 constants).

Note: 2^{nd} order implies there should be 2 parameters in the general solution.

2.1 Reasons for the algorithm

1. Guess a solution of the form $x(t) = e^{rt}$. (Method of optimism)
2. Plug the guess into the DE and do the algebra to see which values of r work:

$$\begin{aligned} & x'' + 8x' + 7x = 0 \\ \Rightarrow\ & r^2 e^{rt} + 8re^{rt} + 7e^{rt} = 0 \\ \Rightarrow\ & \cancel{e^{rt}}(r^2 + 8r + 7) = 0 \quad \text{(safe to cancel } e^{rt} \text{ since it's never 0)} \\ \Rightarrow\ & \underbrace{r^2 + 8r + 7 = 0}_{\text{characteristic equation}} \end{aligned}$$

3. Roots $r = -1, -7$ mean $x_1(t) = e^{-t}, \quad x_2(t) = e^{-7t}$ are both solutions to the DE.
4. Principle of superposition (discussion next) implies

$$x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 e^{-t} + c_2 e^{-7t}$$

are all solutions to the DE.

3 Superposition principle for linear, homogeneous DEs

Superposition principle: For a linear, homogeneous DE, if x_1 and x_2 are solutions, then so are all linear combinations $x = c_1 x_1 + c_2 x_2$, where c_1, c_2 are constants.

– This is straightforward to check by plugging and chugging. (See the Topic 5 notes or today's problems.)

– It's important that the DE is homogeneous. We will have other superposition principles for inhomogeneous, linear DEs.

Problems: Solve (a) $x' + kx = 0$, (b) $x'' + 4x' + 3x = 0$.

Solution: With today's solutions.

4 Complex roots

Example 3. (Complex roots: model solution) Solve $x'' + 4x' + 13x = 0$.

Solution: Characteristic equation: $r^2 + 4r + 13 = 0$.

Roots: $r = -2 \pm 3i$ (quadratic formula).

Basic solutions: $x_1(t) = e^{-2t}\cos(3t)$, $x_2(t) = e^{-2t}\sin(3t)$ (reasons later).

General solution by superposition: $x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t)$.

Note: Exponents = real part of roots. Frequency in cos, sin from imaginary part.

4.1 Reason for solutions with complex roots

1. Have roots $-2 + 3i$, $-2 - 3i$.

2. Our method of optimism guarantees $z_1 = e^{(-2+3i)t}$ and $z_2 = e^{(-2-3i)t}$ are solutions. (They are complex-valued solutions.)

3. The superposition principle says

$$z(t) = d_1 z_1(t) + d_2 z_2(t) = d_1 e^{(-2+3i)t} + d_2 e^{(-2-3i)t}$$

is a solution for all constants d_1, d_2. (d_1, d_2 can be complex!)

4. For a mathematician, $z(t)$ is a fine solution, even though it usually produces a complex number, i.e., $z(t)$ is complex-valued.

5. If x is a physical quantity, we probably want solutions that are real-valued.

6. For special, clever choices of d_1, d_2, $z = d_1 z_1 + d_2 z_2$ is real-valued.

In the problems you will show

$$\left.\begin{aligned} x_1(t) &= \frac{1}{2}e^{(-2+3i)t} + \frac{1}{2}e^{(-2-3i)t} &&= e^{-2t}\cos(3t) \\ x_2(t) &= \frac{1}{2i}e^{(-2+3i)t} - \frac{1}{2i}e^{(-2-3i)t} &&= e^{-2t}\sin(3t) \end{aligned}\right\} \text{real-valued solutions}$$

These were our basic solutions in the model solution. Now, linear combinations of x_1, x_2 give the general solution.

5 Repeated roots

Example 4. Suppose a linear, homogeneous, constant coefficient DE has characteristic roots 2, 8, 3, 3, 3. What is the general solution?

Solution: $x(t) = c_1 e^{2t} + c_2 e^{8t} + c_3 e^{3t} + c_4 t e^{3t} + c_5 t^2 e^{3t}$.

(The justification for this will come in a future topic, after we develop the necessary algebraic techniques.)