Solutions Day 9, R 2/15/2024

Topic 5: Homogeneous, linear, constant coefficient DEs (day 1)

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Problem 1. *Solve $x' + kx$ using the characteristic equation method. Are you surprised by the answer?*

Solution: Characteristic equation: $r + k = 0$

Characteristic roots: $r = -k$

Basic solution: $x_1(t) = e^{-kt}$

General solution: $x(t) = c_1 x_1 = c_1 e^{-kt}$. (First-order implies a one parameter family of solutions.)

Problem 2.

(a) *Solve $x'' + 4x' + 3x = 0$.*

(b) *Find the solution with initial conditions $x(0) = 1, \quad x'(0) = 1$.*

Solution: (a) Characteristic equation: $r^2 + 4r + 3 = 0$

Characteristic roots: $r = -1, -3$

Basic solutions: $x_1(t) = e^{-t}, \quad x_2(t) = e^{-3t}$.

General solution: $x(t) = c_1 x_1 + c_2 x_2 = c_1 e^{-t} + c_2 e^{-3t}$.

(b) Use the initial conditions (IC) to determine the values of c_1, c_2.

$$\begin{array}{lll} x(0) & = c_1 + c_2 & = 1 \\ x'(0) & = -c_1 - 3c_2 & = 1 \end{array} \quad \Rightarrow c_1 = 2,\ c_2 = -1 \quad \Rightarrow \boxed{x(t) = 2e^{-t} - e^{-3t}}.$$

Problem 3. *Give the characteristic equation for each of the following DEs.*

(a) $7x^{(4)} + 3x''' - 5x'' + 2x' + 4x = 0$.

(b) $x'' + x' = 0$.

(c) $a_n x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-1)} + \ldots + a_1 x' + a_0 x = 0$.

(d) $x'' + t^2 x' + 7x = 0$ *(Trick question!)*

Solution: (a) $7r^4 + 3r^3 - 5r^2 + 2r + 4 = 0$.

(b) $r^2 + r = 0$.

(c) $a_n r^n + a_{n-1} r^{n-1} + a_{n-2} r^{n-1} + \ldots + a_1 r + a_0 = 0$.

(d) This is not a linear, constant coefficient, homogeneous DE. It does not have a characteristic equation. That technique will not work to solve this DE.

Problem 4.

(a) *Solve $x'' + x' = 0$.*

Solution: Characteristic equation: $r^2 + r = 0$

Characteristic roots: $r = 0, -1$

Basic solutions: $x_1(t) = e^{0\cdot t} = 1$, $x_2(t) = e^{-t}$.

General solution: $x(t) = c_1x_1 + c_2x_2 = c_1 + c_2e^{-t}$.

(b) *Solve $x'' + 4x = 0$.*

Solution: Characteristic equation: $r^2 + 4 = 0$ (Not $r^2 + 4r = 0$!)

Characteristic roots: $r = \pm 2i$

Basic solutions: $x_1(t) = \cos(2t)$, $x_2(t) = \sin(2t)$.

General solution: $x(t) = c_1x_1 + c_2x_2 = c_1\cos(2t) + c_2\sin(2t)$.

Problem 5. *A constant coefficient, linear, homogeneous DE has characteristic roots*

$$-1,\ -2,\ -2,\ -2,\ -3 \pm 4i,\ -5 \pm 6i,\ -5 \pm 6i.$$

(a) *What is the order of the DE? (Notice the $\pm$ in the list of roots.)*

Solution: 10 roots implies it is a 10th order DE.

(b) *What is the general, real-valued solution.*

Solution: The 10 roots give 10 basic solutions:

$$\begin{array}{llll}
x_1 = e^{-t} & & & \\
x_2 = e^{-2t} & x_3 = te^{-2t} & x_4 = t^2e^{-et} & \\
x_5 = e^{-3t}\cos(4t) & x_6 = e^{-3t}\sin(4t) & & \\
x_7 = e^{-5t}\cos(6t) & x_8 = e^{-5t}\sin(6t) & x_9 = te^{-5t}\cos(6t) & x_{10} = te^{-5t}\sin(6t)
\end{array}$$

The general solution is

$$x(t) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6 + c_7x_7 + c_8x_8 + c_9x_9 + c_{10}x_{10}.$$

Problem 6. *State and verify the superposition principle for $mx'' + bx' + kx = 0$, (m, b, k constants).*

Solution: Superposition principle for linear, homogeneous DEs:

If x_1 and x_2 are solutions to the DE, then so are all linear combinations $x = c_1x_1 + c_2x_2$.

Proof. Plug x into the DE and then chug through the algebra to show that x is a solution.

$$\begin{aligned}
mx'' + bx' + kx &= m(c_1x_1 + c_2x_2)'' + b(c_1x_1 + c_2x_2)' + k(c_1x_1 + c_2x_2) \\
&= c_1mx_1'' + c_2mx_2'' + c_1bx_1' + c_2bx_2' + c_1kx_1 + c_2kx_2 \\
&= c_1\underbrace{(mx_1'' + bx_1' + kx_1)}_{\substack{\text{0 by assumption that} \\ x_1 \text{ is a solution}}} + c_2\underbrace{(mx_2'' + bx_2' + kx_2)}_{\substack{\text{0 by assumption that} \\ x_2 \text{ is a solution}}} \\
&= 0 \quad \blacksquare
\end{aligned}$$