Formula Sheet for Final Quiz

Variation of parameters formula

$$x(t) = x_h(t) \int \frac{q(t)}{x_h(t)}\, dt + C x_h(t).$$

Fourier Series for $f(t)$, periodic with period $2L$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\, \cos\left(\frac{\pi}{L} nt\right)\, dt, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\, \sin\left(\frac{\pi}{L} nt\right)\, dt$$

Period 2π square wave:

$$\mathrm{sq}(t) = \begin{cases} -1 \text{ on } -\pi < t < 0 \\ 1 \text{ on } 0 < t < \pi \end{cases} \quad = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}$$

Period 2π triangle wave:

$$\mathrm{tri}(t) = |t| \text{ on } -\pi < t < \pi \;=\; \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}$$

Exponential and Sinusoidal Response Formulas

$P(D)y = e^{at}$ has particular solution:

$$y_p(t) = \begin{cases} \dfrac{e^{at}}{P(a)} & \text{if } P(a) \neq 0 \\ \dfrac{te^{at}}{P'(a)} & \text{if } P(a) = 0 \text{ and } P'(a) \neq 0 \end{cases}$$

$P(D)y = \cos(\omega t)$ has particular solution

$$y_p(t) = \begin{cases} \dfrac{\cos(\omega t - \phi)}{|P(i\omega)|} & \text{where } \phi = \mathrm{Arg}(P(i\omega)) \quad \text{if } |P(i\omega)| \neq 0 \\ \dfrac{t\cos(\omega t - \phi)}{|P'(i\omega)|} & \text{where } \phi = \mathrm{Arg}(P'(i\omega)) \quad \text{if } P(i\omega) = 0 \text{ and } P'(i\omega) \neq 0 \end{cases}$$

(Integral table on next page)

Integrals

For n a positive integrer

$$\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n} \text{ if } n \neq 0.$$

$$\int_0^\pi t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$$

$$\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \text{for } n \text{ odd} \\ \dfrac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases}$$

$$\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2} \text{ if } n \neq 0.$$

For arbitrary ω

$$\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$$

$$\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$$

$$\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$$

$$\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$$

ES.1803 Differential Equations

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