18.03 Completeness of Fourier Expansion

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1 Introduction

In Topic 21 we stated, without proof, the Fourier theorem, which says that a period $2L$ function can be written as a Fourier series with the coefficients given by some integral formulas. In Topic 22 we used the orthogonality relations to show that a periodic function with a Fourier series must have the coefficients given by the integral formulas.

This only proves half the Fourier theorem. The other half is to show that every periodic function must have a Fourier series. That is, we must show that a continuous periodic function actually equals its Fourier series. This is the goal of this note.

To keep things simple, we will only work with period 2π functions. The generalization to period $2L$ functions is just a change of variable.

That every period 2π function has a Fourier series is called completeness. This is because it says the set of functions $\{\cos(nt), \sin(nt) \,|\, n = 0, 1, 2, ...\}$ form a complete set of basis functions. That is, you don't need any more functions to express every period 2π function as a linear combination.

2 Competeness theorem

Theorem: (Completeness theorem)

A continuous periodic function $f(t)$ equals its Fourier series.

Notes: 1. We will gloss over some analytic issues like convergence, they are not too hard in the current context.

2. This theorem can be proved as a simple consequence of a theorem from analysis called the Stone-Weierstass theorem. We will not resort to that.

3 Proof of the completeness theorem

For concreteness we will assume $f(t)$ has period 2π. The generalization to period $2L$ functions only requires a change of variable.

Before we start the proof proper, we need some preliminary notions.

3.1 Convolution

Assume f and g have period 2π. The the convolution of f with g is defined by

$$f * g(t) = \int_{-\pi}^{\pi} f(u)g(t-u)\,du.$$

It is not hard to show that this is commutative, i.e., $f * g = g * f$. We do that at the end of this note.

3.2 Periodic delta function and its approximation

The periodic δ function (or period 2π impulse train) is defined by

$$\tilde{\delta}(t) = ... + \delta(t + 4\pi) + \delta(t + 2\pi) + \delta(t) + \delta(t - 2\pi) + ...$$

Claim: If $f(t)$ is continuous and periodic (period 2π) then $\tilde{\delta} * f(t) = f(t)$.

Proof: In the interval $[-\pi, \pi]$ the only non-zero term in the sum defining $\tilde{\delta}(t)$ is $\delta(t)$. So,

$$\tilde{\delta} * f(t) = \int_{-\pi}^{\pi} \tilde{\delta}(u) f(t-u)\, du = \int_{-\pi}^{\pi} \delta(u) f(t-u)\, du = f(t).$$

$\tilde{\delta}(t)$ is called the convolutional identity.

Now, consider the function $h(t) = \left(\frac{1+\cos(t)}{2}\right)$.

(i) h is periodic with period 2π

(ii) $h(0) = 1$

(iii) As t goes from 0 to π (or $-\pi$) $h(t)$ decreases to 0.

Consequently for large k the graph of $h(t)^k$ between $-\pi$ and π is nearly a spike of unit height at the origin. With this in mind we define

$$h_k(t) = c_k \left(\frac{1+\cos(t)}{2}\right)^k$$

where c_k is chosen so that $\int_{-\pi}^{\pi} h_k(t)\, dt = 1$.

As k grows the graph gets thinner and spikier (in order for this to have area 1 we must have c_k growing larger). Since the area is always 1, this shows

$$\lim_{k \to 0} h_k(t) = \tilde{\delta}(t).$$

Because of this limit, we say the sequence $h_k(t)$ is an approximation of the (convolutional) identity.

Notice that h_k is a linear combination of powers of $\cos(t)$. Also recall that powers of $\cos(t)$ can all be written as linear combinations of terms of the form $\sin(nt)$ and $\cos(nt)$. (You can easily show this using Euler's formula.) Combining these statements, we have:

h_k is a linear combination terms of the form $\sin(nt)$ and $\cos(nt)$.

3.3 Proof of the completeness theorem

Denote the Fourier series of $f(t)$ by $f_1(t)$. We know

$$f_1(t) = \frac{a_0}{2} + \sum a_n \cos(nt) + \sum b_n \sin(nt)$$

where

$$a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt, \text{ and } b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt.$$

The orthogonality relations guarantee that f_1 gives the same coefficients. That is,

$$a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f_1(t)\cos(nt)\,dt, \text{ and } b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f_1(t)\sin(nt)\,dt.$$

Our goal is to show that $f(t) = f_1(t)$, or equivalently $g(t) = f(t) - f_1(t) = 0$. To do this, first note that

$$\int_{-\pi}^{\pi} g(t)\sin(nt)\,dt = \int_{-\pi}^{\pi} (f(t) - f_1(t))\sin(nt)\,dt = b_n - b_n = 0.$$

Likewise,

$$\int_{-\pi}^{\pi} g(t)\cos(nt)\,dt = \int_{-\pi}^{\pi} (f(t) - f_1(t))\cos(nt)\,dt = a_n - a_n = 0.$$

Since $h_k(t)$ is just a sum of sines and cosines, this shows

$$h_k * g(t) = \int_{-\pi}^{\pi} h_k(t-u)g(u)\,du = 0.$$

So,

$$\lim_{k\to\infty} h_k * g(t) = 0.$$

But this limit is also $\tilde{\delta} * g(t) = g(t)$. That is, we have shown $g(t) = 0$. ■

4 Appendix: Proof that $f * g = g * f$

This is just a change of variable: Assume $f(t)$ and $g(t)$ are period 2π functions. Then

$$\begin{aligned}
(f * g)(t) &= \int_{-\pi}^{\pi} f(u)g(t-u)\,du \\
&= \int_{t-\pi}^{t+\pi} f(t-v)g(v)\,dv \qquad \text{(change of variable } u = t - v) \\
&= \int_{-\pi}^{\pi} g(v)f(t-v)\,dv \\
&= (g * f)(t) \qquad \blacksquare.
\end{aligned}$$

The third equality follows because, for periodic functions, the integrals over any full period are all the same.