18.03 Gibbs' Phenomenon

Jeremy Orloff

Notation

To keep the notation simple, we will work with period 2π functions. Also, we will assume that any jump discontinuties happen at $t = 0$. Generalizing our arguments to other periods or points of discontinuity is reasonably straightforward.

Truncated Fourier series

Suppose $f(t)$ is a period 2π function with Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt).$$

The truncated Fourier series for f is defined by summing only a finite number of terms. That is, it is defined by

$$S_{N,f}(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt).$$

Gibbs' Phenonemon says that the truncated Fourier series near a jump discontinuity overshoots the jump by about 9% of the size of the jump.

Gibbs' phenomenon for a square wave

We start with the key example of a period 2π square wave: $f_1(t) = \begin{cases} 0 & \text{for } -\pi < t < 0 \\ 1 & \text{for } 0 < t < \pi. \end{cases}$

Since $f_1(t) = \frac{1}{2}(1 + \text{sq}(t))$, where $\text{sq}(t)$ is our usual odd, period 2π square wave, we have

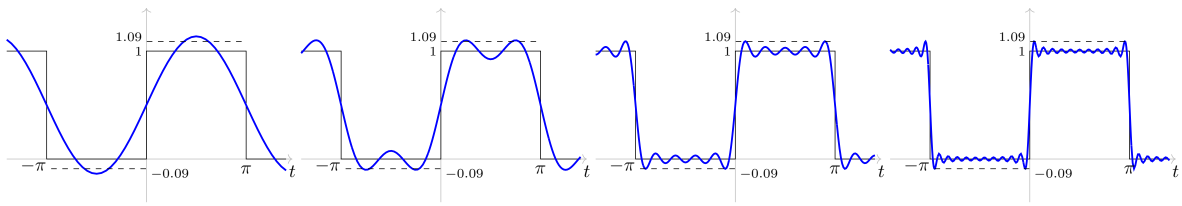
$$f_1(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n} \tag{1}$$

The trunctated Fourier series is given by

$$S_{2N-1,f_1}(t) = \frac{1}{2} + \frac{2}{\pi} \left(\sin(t) + \frac{\sin(3t)}{t} + ... + \frac{\sin((2N-1)t)}{2N-1} \right) \tag{2}$$

Gibbs' phenomenon for $f_1(t)$: The maximum value of $S_{2N-1,f_1}(t)$ always overshoots $f_1(t)$ by about 9%. That is, the maximum value is about 1.089 and never disappears, no matter how large N becomes. As N goes to infinity, the point where the maximum overshoot occurs goes to the jump discontinuity.

The figures below show $f_1(t)$ and its truncated Fourier series for several values of N. Notice that the overshoot moves towards the jump, but stays at about 1.09 (or -0.09 on the bottom).



Sum up to $n = 1$ Sum up to $n = 3$ Sum up to $n = 9$ Sum up to $n = 21$

The proof of Gibbs' phenomenon for $f_1(t)$ is an elaborate and somewhat tricky calculus exercise. We'll show it in a number of steps.

Step 1: We claim $S'_{2N-1,f_1}(t) = \dfrac{1}{\pi} \cdot \dfrac{\sin(2Nt)}{\sin(t)}$.

Proof: Differentiating Equation 2, we get

$$S'_{2N-1,f_1}(t) = \frac{2}{\pi}(\cos(t) + \cos(3t) + ... + \cos((2N-1)t)).$$

Recall the formulas for $\cos(t)$ and $\sin(t)$ in terms of complex exponentials:

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}, \qquad \sin(t) = \frac{e^{it} - e^{-it}}{2i}.$$

Use these to rewrite the formula for S'_{2N-1,f_1}:

$$S'_{2N-1,f_1}(t) = \frac{2}{\pi}\left(\frac{e^{(-2N+1)it} + e^{(-2N+3)it} + ... e^{(2N-3)it} + e^{(2N-1)it}}{2}\right).$$

This is a geometric series with ratio e^{2it}. It has sum

$$S'_{2N-1,f_1}(t) = \frac{1}{\pi} \cdot \frac{e^{(-2N+1)it} - e^{(2N+1)it}}{1 - e^{2it}}.$$

Multiply top and bottom by e^{-it} and use the formula for $\sin(t)$ in terms of complex exponentials to get

$$S'_{2N-1,f_1}(t) = \frac{1}{\pi} \cdot \frac{e^{(-2N)it} - e^{(2N)it}}{e^{-it} - e^{it}} = \frac{1}{\pi} \cdot \frac{\sin(2Nt)}{\sin(t)}. \quad \text{QED}$$

Step 2: Find the first positive maximum of $S_{2N-1,f_1}(t)$.

The formula for $S'_N(t)$ shows that $S_{2N-1,f_1}(t)$ has critical points at multiples of $\dfrac{\pi}{2N}$. So, the first positive critical point is at $t = \dfrac{\pi}{2N}$.

Since, $S_{2N-1,f_1}(0) = \frac{1}{2}$ and all the terms in the sum for $S_{2N-1,f_1}(\pi/2N)$ are positive we conclude that $t = \pi/2N$ is a local maximum (it is, in fact, the absolute maximum).

(Alternatively, you could use the second derivative to show this is a relative maximum. Or you could use L'Hospital's rule to see that $S'_N(0) = 2N > 0$, which implies that $S_{2N-1,f_1}(t)$ is rising towards a maximum.)

Step 3: Estimate the maximum value of $S_{2N-1,f_1}(t)$, i.e., estimate $S_{2N-1,f_1}(\pi/2N)$.

First we manipulate the series for $S_{2N-1,f_1}(\pi/2N)$:

$$\begin{aligned} S_{2N-1,f_1}\left(\frac{\pi}{2N}\right) &= \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin(\pi/2N)}{1} + \frac{\sin(3\pi/2N)}{3} + ... + \frac{\sin((2N-1)\pi/2N)}{2N-1}\right) \\ &= \frac{1}{2} + \frac{1}{\pi}\cdot\frac{\pi}{N}\left(\frac{\sin(\pi/2N)}{\pi/2N} + \frac{\sin(3\pi/2N)}{3\pi/2N} + ... + \frac{\sin((2N-1)\pi/2N)}{(2N-1)\pi/2N}\right) \end{aligned}$$

This last is a Riemman sum (using midpoints) for

$$\frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin(t)}{t}\,dt, \quad \text{with } \Delta t = \frac{\pi}{N}.$$

Since $\Delta t \to 0$ as $N \to \infty$ we get

$$\lim_{N\to\infty} S_{2N-1,f_1}\left(\frac{\pi}{2N}\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin(t)}{t}\,dt.$$

So, $S_{2N-1,f_1}\left(\frac{\pi}{2N}\right) \approx \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin(t)}{t}\,dt.$

All that's left is to estimate the value of the integral. For this we integrate the power series for $\frac{\sin(t)}{t}$. We have

$$\frac{\sin(t)}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + ...$$

Which gives

$$S_{2N-1,f_1}\left(\frac{\pi}{2N}\right) \approx \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin(t)}{t}\,dt = \frac{1}{2} + \frac{1}{\pi}\left(\pi - \frac{\pi^3}{3\cdot 3!} + \frac{\pi^5}{5\cdot 5!} - \frac{\pi^7}{7\cdot 7!} + ...\right)$$

This series converges very rapidly and, after six terms, we get the value 1.089 correct to 3 decimal places.

We have seen that, as N gets large, the maximum value of $S_{2N-1,f_1}(t)$ becomes approximately 1.089. That is, it overshoots the correct value by about 0.089, i.e., by about 9% of the jump from 0 to 1.

Gibbs' phenomenon for a function with jump discontinuities

Now let $f(t)$ be a piecewise smooth, period 2π function with jump discontinuties. To avoid too much notation, let's assume that there is a jump discontinuity of height h at $t = 0$.

Gibbs' phenomenon: At $t = \pi/2N$, the truncated Fourier series overshoot the correct value of f by about $0.089\,h$. That is, by about 9% of the jump. In symbols,

$$S_{2N-1,f}\left(\frac{\pi}{2N}\right) \approx f\left(\frac{\pi}{2N}\right) + 0.089\,h$$

Proof: Let $g(t) = f(t) - h\,f_1(t)$. Since $f(t)$ has a jump of h at $t = 0$ and $-h\,f_1(t)$ has a jump of $-h$, $g(t)$ has no jump, i.e., it's continuous at $t = 0$.

Since $g(t)$ is continuous, at $t = 0$, its Fourier series converges to $g(0)$. That is $S_{N,g}(t) \approx g(t)$, for t near 0.

Now $f(t) = g(t) + h\, f_1(t)$. So,

$$
\begin{aligned}
S_{2N-1,f}\left(\frac{\pi}{2N}\right) &= S_{2N-1,g}\left(\frac{\pi}{2N}\right) + hS_{2N-1,f_1}\left(\frac{\pi}{2N}\right) \\
&\approx g\left(\frac{\pi}{2N}\right) + 1.089\, h && \text{(by our known overshoot for } f_1\text{)} \\
&= f\left(\frac{\pi}{2N}\right) - h\, f_1\left(\frac{\pi}{2N}\right) + 1.089\, h && \text{(by the definition of } g(t)\text{)} \\
&= f\left(\frac{\pi}{2N}\right) - h + 1.089\, h && \left(\text{since } f_1\left(\frac{\pi}{2N}\right) = 1\right) \\
&= f\left(\frac{\pi}{2N}\right) + 0.089\, h. && \blacksquare
\end{aligned}
$$