ES.1803: Complexification vs. Inverse Euler's Formula

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1 Introduction

We have two formulas for $\cos(t)$:

1. Taking the real part: $\cos(t) = \text{Re}(e^{it})$

2. Inverse Euler formula: $\cos(t) = \dfrac{e^{it} + e^{-it}}{2}$.

Of course, these two formulas are intimitely related. But, sometimes students confuse when to use one over the other. This note will give a set of examples designed to get at the differences.

2 Complex replacement

When we use complex replacement, we replace $\cos(\omega t)$ by $e^{i\omega t}$. Then at the end, we usually need to take the real part to undo the complex replacement.

Example 1. Complex replacement (complexification). Let $P(D) = 8D^2 + 8D + 7I$. Solve $P(D)x = \cos(2t)$.

Here, we replace x by $z = x + iy$ and $\sin(2t)$ by $e^{2it} = \cos(2t) + i\sin(2t)$. So,

$$P(D)z = e^{2it}, \qquad z = \text{Re}(x).$$

Now use the ERF to find $z_p(t) = \dfrac{e^{2it}}{P(2i)} = \dfrac{e^{2it}}{3+16i}$.

In polar form: $3 + 16i = \sqrt{265}\, e^{i\phi}$, where $\phi = \text{Arg}(3 + 16i)$. So, $z_p(t) = \dfrac{e^{i(2t-\phi)}}{\sqrt{265}}$.

The last step is to *undo the complexification* by taking the imaginary part:

$$x_p(t) = \text{Re}(z_p) = \frac{\cos(2t - \phi)}{\sqrt{265}}.$$

We can say, if you 'complexified' to start, you should 'uncomplexify' at the end.

3 Euler's formula

Example 2. (Euler's formula)

Suppose we want to solve $P(D)x = 0$ and we find characteristic roots $-1 \pm 2i$. Two roots should give us 2 independent solutions. If we want real-valued solutions, these roots give us

$$x_1(t) = e^{-t}\cos(2t) \quad \text{and} \quad x_2(t) = e^{-t}\sin(2t).$$

You should recall, these are the real and *imaginary* parts of the complex exponential solution $z(t) = e^{(-1+2i)t}$.

The mistake I often see is that someone will want the **real-valued solution** so they will take just the real part of $z(t)$.

Remember: both the real and imaginary parts are real-valued and linearity says they are both solutions!

4 Inverse Euler's formula

Example 3. (Inverse Euler's formula)

If we are given something like $\int \cos^2(t)\,dt$, then it will not work to replace $\cos^2(t)$ by $(e^{it})^2$, because $\cos^2(t)$ is not the real part of $(e^{it})^2$.

Instead, we can use the inverse Euler formula: $\cos(t) = (e^{it} + e^{-ti})/2$.

This gives

$$\int \cos^2(t)\,dt = \int \left(\frac{e^{it}+e^{-it}}{2}\right)^2 dt = \int \frac{e^{2it}}{4} + \frac{1}{2} + \frac{e^{-2it}}{4}\,dt = \frac{e^{2it}}{8i} + \frac{t}{2} - \frac{e^{-2it}}{8i} = \frac{\sin(2t)}{4} + \frac{t}{2}.$$

The last equality follows from the inverse Euler formula for $\sin(2t)$:

$$\sin(2t) = \frac{e^{2it} - e^{-2it}}{2i}$$

Note: We never complexified, so we didn't have to take the real part at the end. In fact, the algebra just worked out to give us a real-valued solution!

We can say that if you didn't complexify at the start, you don't need to uncomplexify at the end.