18.03 Power Series Techniques

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1 Introduction

Using the method of optimism to guess a power series solution allows us to solve a wide range of problems. We will give a hint at the power of this technique with a series of examples.

Our main reference for this note is Chapter 3 of the text by Edwards and Penney:

Edwards, C. and Penney, D, *Elementary Differential Equations with Boundary Value Problems* (fifth ed.). Upper Saddle River, N.J.: Prentice Hall, 2004.

We will consider the second-order linear DE:

$$y'' + P(x)y' + Q(x) = 0$$

We will do all our work around $x = 0$, it is easy to translate this to $x = a$. Also, we will not worry about radius of convergence or other analytic issues.

2 Ordinary points:

If $P(x)$ and $Q(x)$ are analytic (have a convergent power series) around $x = 0$ then 0 is called an *ordinary point*

Example 1. Solve $y'' + y = 0$.

Solution: Try a power series solution: $y(x) = \sum_0^\infty a_n x^n$.

So, $y'' = \sum_2^\infty n(n-1)a_n x^{n-2} = \sum_0^\infty (n+2)(n+1)a_{n+2}x^n$.

Thus, $y'' + y = 0 \quad \Rightarrow \sum_0^\infty [(n+2)(n+1)a_{n+2} + a_n]x^n = 0$.

For the series to equal 0, we need the coefficient of x^n to be 0, for every n. That is,

$$(n+2)(n+1)a_{n+2} + a_n = 0 \quad \Rightarrow a_{n+2} = -\frac{a_n}{(n+2)(n+1)}.$$

This last equation is called the recurrence relation for the coefficients a_n.

Pick an arbitrary a_0. Looking at the recurrence relations we have

$n = 0$: $\quad a_2 = -\frac{a_0}{2 \cdot 1}$.

$n = 2$: $\quad a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_0}{4!}$.

$n = 4$: $\quad a_6 = -\frac{a_4}{6 \cdot 5} = -\frac{a_0}{6!}$

Continuing: $a_{2n} = (-1)^n \dfrac{a_0}{(2n)!}$.

Likewise, pick an arbitrary a_1:

$n = 1$: $a_3 = -\dfrac{a_1}{3 \cdot 2}$.

Continuing: $a_{2n+1} = (-1)^n \dfrac{a_1}{(2n+1)!}$.

Thus, $y = a_0 \left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - ...\right) + a_1 \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - ...\right) = a_0 \cos(x) + a_1 \sin(x)$.

Of course, this is the answer we knew we should get for this DE.

Example 2. Solve $y' = x^2 y$.

Solution: Try $y = \sum_0^\infty a_n x^n$.

So, $y' = \sum_0^\infty n a_n x^{n-1} = a_1 + 2a_2 x + \sum_2^\infty (n+1) a_{n+1} x^n$.

Substitution: $y' - x^2 y = 0 \quad \Rightarrow a_1 + 2a_2 x + \sum_2^\infty [(n+1)a_{n+1} - a_{n-2}] x^n = 0$.

All the coefficients are 0 gives the recurrence relations:

$$a_1 = 0,\ a_2 = 0,\ (n+1)a_{n+1} - a_{n-2} = 0 \qquad \Rightarrow a_{n+1} = \frac{a_{n-2}}{n+1}.$$

Pick an arbitrary a_0. The recurrence relations imply $a_3 = \dfrac{a_0}{3}$, $a_6 = \dfrac{a_3}{6} = \dfrac{a_0}{6 \cdot 3} = \dfrac{a_0}{3^2 \cdot 2!}$.

Continuing: $a_9 = \dfrac{a_6}{9} = \dfrac{a_0}{3^3 \cdot 3!}$, $a_{12} = \dfrac{a_0}{3^4 \cdot 4!}$, ..., $a_{3n} = \dfrac{a_0}{3^n \cdot n!}$.

$a_1 = 0 \Rightarrow a_4 = 0 \Rightarrow a_7 = 0 ...$

$a_2 = 0 \Rightarrow a_5 = 0 \Rightarrow a_8 = 0 ...$

Thus, $y = a_0 \sum_0^\infty \dfrac{x^{3n}}{3^n \cdot n!} = a_0 e^{x^3/3}$.

Of course, this equation is separable and you can check our answer using separation of variables.

3 Regular Singular Points

Consider the DE $y'' + \dfrac{p(x)}{x} y' + \dfrac{q(x)}{x^2} y = 0$. Because the coefficients are discontinuous at $x = 0$ we call 0 a singular point. If $p(x)$ and $q(x)$ are analytic (in particular if they are polynomials) then $x = 0$ is a called a regular singular point.

In this case, we only look for solutions for $x > 0$.

Example 3. Solve $y'' + \dfrac{p_0}{x} y' + \dfrac{q_0}{x^2} y = 0$, where p_0, q_0 are constants.

Solution: Because p_0 and q_0 are constants, it turns out, it will work if we guess a solution of the form $y = x^r$.

Try $y = x^r$.

Substitution: $r(r-1)x^{r-2} + p_0 r x^{r-2} + q_0 x^{r-2} = 0$.

This implies

$$r(r-1) + p_0 r + q_0 = 0. \tag{1}$$

This is called the *indicial equation.*

If the roots are real and different we have two solutions.

If the roots are complex, say $r = a \pm ib$, we have the solution

$$z(x) = x^{a+ib} = x^a e^{ib\log x} = x^a(\cos(b\log x) + i\sin(b\log x)).$$

Likewise for x^{a-ib}. And, just like CC homogeneous linear DEs, this means we have two real solutions

$$y_1(x) = x^a\cos(b\log x) \quad \text{and} \quad y_2(x) = x^a\sin(b\log x).$$

From now on we will focus on equations with real roots.

3.1 Bessel's equation

Example 4. (Bessel equation of order m)

Solve $x^2y'' + xy' + (x^2 - m^2)y = 0$. (Note, $x = 0$ is a regular singular point.)

Solution: A simple $y = x^r$ won't work, instead we need the Frobenius solution. That is x^r times a power series:

$$y = x^r\sum_0^\infty a_n x^n = \sum_0^\infty a_n x^{n+r}$$

Note, this is not a power series if r is not an integer. Also, without loss of generality we can require $a_0 \neq 0$. We get:

$$x^2y'' = \sum_0^\infty (n+r)(n+r-1)a_n x^{n+r} = r(r-1)a_0x^r + (r+1)ra_1x^{r+1} + \sum_2^\infty (n+r)(n+r-1)a_n x^{n+r}$$

$$xy' = \sum_0^\infty (n+r)a_n x^{n+r} = ra_0x^r + (r+1)a_1x^{r+1} + \sum_2^\infty (n+r)a_n x^{n+r}$$

$$x^2y = \sum_0^\infty a_n x^{n+r+2} = \sum_2^\infty a_{n-2}x^{n+r}$$

$$m^2y = \sum_0^\infty m^2a_n x^{n+r} = m^2a_0x^r + m^2a_1x^{r+1} + \sum_2^\infty m^2a_n x^{n+r}$$

Substituting into the DE we get

$$[r(r-1) + r - m^2]a_0x^r + [(r+1)r + (r+1) - m^2]a_1x^{r+1}$$
$$+ \sum_2^\infty [((n+r)(n+r-1) + (n+r) - m^2)a_n + a_{n-2}]x^{n+r} = 0.$$

Cleaning things up and setting coefficients to 0, we get

$$(r^2 - m^2)a_0 = 0, \quad ((r+1)^2 - m^2)a_1 = 0, \quad ((n+r)^2 - m^2)a_n + a_{n-2} = 0.$$

Since we assume $a_0 \neq 0$, The first equation, $(r^2 - m^2)a_0 = 0$ implies $r^2 - m^2 = 0$, i.e. this determines the possible values of r. The equation $r^2 - m^2 = 0$ is called the indicial equation.

Example 5. Find the solution for $m = 1/3$ in the above example.

Solution: Take $a_0 \neq 0$ arbitrary. The indicial equation $r^2 - m^2 = 0$ implies $r = \pm 1/3$.

Take $r = 1/3$: The equation $((r+1)^2 - m^2)a_1 = 0$ implies $a_1 = 0$.

The general recurrence relation is $a_n = -\dfrac{a_{n-2}}{(n+r)^2 - m^2}$.

Using $a_1 = 0$, this implies $a_3 = a_5 = a_7 = ... = 0$.

Starting with a_0, the recurrence relation implies $a_2 = -\dfrac{a_0}{(2+1/3)^2 - 1/9}$, etc. That is, it gives all the values a_2, a_4, a_6, ...

Likewise for $r = -1/3$.

This gives us two independent solutions to the linear DE.

Example 6. Take $m = 1/2$ and find the solution.

Solution: The indicial equation $r^2 - 1/4 = 0$ implies $r = \pm 1/2$.

Take $r = -1/2$: Pick an arbitrary value for a_0. The recurrence relation then implies

$$a_2 = -\frac{4a_0}{3^2 - 1}, \quad a_4 = -\frac{4a_2}{7^2 - 1}, \quad \ldots$$

The equation $((r+1)^2 - m^2)a_1 = 0$ shows that a_1 can also be arbitrary. The recurrence equation then determines

$$a_3 = -4a_1/(5^2 - 1), \quad a_5 = \ldots, \quad a_7 = \ldots$$

So, the solution $y(x) = \sum a_n x^{n+r}$ has two arbitrary constants, a_0, a_1.

Note, if we take $r = 1/2$, we get the same solution as with $r = -1/2$ when $a_0 = 0$ and a_1 is arbitrary.

Example 7. Repeat the above examples with $m = 1$.

Solution: Indicial equation: $r^2 - m^2 = 0 \quad \Rightarrow r = \pm 1$.

Take $r = 1$, $a_0 \neq 0$: $\Rightarrow a_2 = -\dfrac{a_0}{3^2 - 1}$, $a_4 = -\dfrac{a_2}{5^2 - 1}$ ...

The equation $((r+1)^2 - m^2)a_1 = 0 \quad \Rightarrow a_1 = 0$.

The recurrence relation the implies $a_3 = 0, \quad a_5 = 0, \ldots.$

Thus, we have a solution $y_1(x) = \displaystyle\sum_{n=0}^{\infty} a_{2n} x^{2n+1}$.

Take $r = -1$ and $a_0 \neq 0$: The recurrence equation $a_2 = -\dfrac{a_0}{(2+r)^2 - 1}$ doesn't work because the denominator is 0.

There is a trick, called reduction of order to deal with this.

Try $y_2(x) = v(x)y_1(x)$. Substituting y_2 into the DE produces the following

$$y_1 v'' + (2y_1' + 1/x)v' = 0.$$

Now, let $u = v' \quad \Rightarrow y_1 u' + (2y_1' + 1/x)u = 0$. This is a first-order linear, homogeneous DE, i.e., we have reduced the order.

This leads to, $\;y_2(x) = C_1 y_1(x) \ln x + x^{-1} \sum_{n=0}^{\infty} b_n x^n$.

Substitution into the gives relations for C_1 and b_n.

Example 8. Repeat the above examples with $m = 0$.

Solution: Indicial equation: $\quad r^2 = 0 \Rightarrow$ repeated roots.

Take $r = 0$, $a_0 \neq 0$: $\quad$ As in the previous examples, this gives us a solution $y_1(x)$.

Reduction of order leads to $\;y_2(x) = y_1 \ln x + \sum_0^{\infty} b_n x^n$. Substitution gives relations for b_n.

Note: In general, if the roots differ by an integer the recurrence relations can sometimes run into trouble. You can read Sections 3.3 and 3.4 of Edwards and Penney or look online for details.