Review of entire semester, Spring 2024

This is a large set of problems covering all the topics. Most are taken from the problem section worksheets.

Topic 1. Modeling; separable DEs

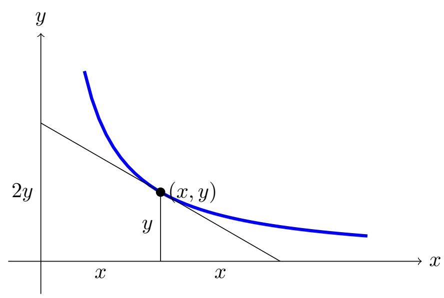
Problem 1. *(Here's the second geometry example in the notes for Topic 1.)*

$y = y(x)$ is a curve in the first quadrant. The part of the tangent line in the first quadrant is bisected by the point of tangency. Find and solve the DE for this curve.

Solution: From the picture: the slope of the tangent $= \dfrac{dy}{dx} = \dfrac{-y}{x}$.

Separate variables: $\dfrac{dy}{y} = -\dfrac{dx}{x}$.

Integrate: $\ln|y| = -\ln|x| + C \;\Rightarrow\; \boxed{y = C/x.}$



Problem 2. *Consider the family of all lines whose y-intercept is twice the slope.*

(a) *Find a DE which has this family as its solutions.*

Solution: The lines are $y = mx + 2m = m(x+2)$. The key here is to end up with a DE in x and y that doesn't explicitly use the slope m. (The slope will be determined by the choice of C in the solution.) We have two different ways of finding m, so

$$\frac{dy}{dx} = m = \frac{y}{x+2}.$$

(b) *Find the orthogonal trajectories to the curves in Part (a). That is, find a family of functions whose graphs intersect all the lines in Part (a) orthogonally.*

Solution: Curves intersect orthogonally if their slopes (at points of intersection) are negative reciprocals. Taking the DE in Part (a) we get

$$\frac{dy}{dx} = -\frac{x+2}{y}.$$

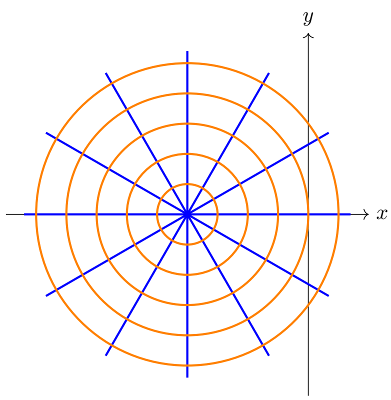
This is separable: $y\,dy = -(x+2)\,dx$.

Integrating: $y^2/2 = -(x+2)^2/2 + C$.

(Changing the meaning of C slightly.) We have $y^2 + (x+2)^2 = C$. This is a circle with center at (-2,0).

(c) *Sketch both families.*

Solution: Note that all the lines go through the point $(-2, 0)$, which is the center of the orthogonal circles.



Orthogonal lines and circles. The center is at $(-2, 0)$

Problem 3. *You deposit money in a bank at the rate of \$1000/year. The money earns (continuous) 8% interest. Construct a DE to model the amount of money in the bank as a function of time; then solve the DE. Assume that at time 0 there is no money in the bank.*

Solution: Let $x(t)$ = amount in bank.

Quick answer: bank accounts have exponential growth, \$1000/year is the input $\Rightarrow \dfrac{dx}{dt} = 0.08\,x + 1000$.

Slower answer: Over a small time Δt, the amount x, the interest rate and the deposit rate are all approximately constant. So, $\Delta x \approx 0.08\,x\,\Delta t + 1000\,\Delta t \;\Rightarrow\; \frac{\Delta x}{\Delta t} \approx 0.08x + 1000$ $\Rightarrow \dfrac{dx}{dt} = 0.08\,x + 1000$.

This is separable and first-order linear. You should solve this carefully, labeling each step. Use whichever technique you prefer.

The solution is $x(t) = 12500(e^{0.08t} - 1)$.

Topic 2. Linear DEs

Problem 4. (Linear homogeneous)
(a) *Solve $y' + ky = 0$.*

Solution: This is our standard exponential decay equation: $y' = -ky$. So, $y(t) = Ce^{-kt}$.

(b) *Solve $y' + ty = 0$.*

Solution: This is separable: $\dfrac{dy}{y} = -t\,dt \quad \Rightarrow y(t) = Ce^{-t^2/2}$.

Problem 5. *Solve $y' + ty = t^3$. (Hint: use Part (b) of the previous problem.*

Solution: This is first-order linear, but not constant coefficient. We have the homogeneous solution from the previous problem: $y_h(t) = e^{-t^2/2}$. Therefore, using the variation of parameters formula:

$$\begin{aligned} y(t) &= y_h(t)\left[\int \frac{t^3}{y_h(t)}\,dt + C\right] \\ &= e^{-t^2/2}\left[\int t^3 e^{t^2/2}\,dt + C\right] = e^{-t^2/2}\left[t^2e^{t^2/2} - 2e^{t^2/2} + C\right] = \boxed{t^2 - 2 + Ce^{-t^2/2}.} \end{aligned}$$

(The integral was done by parts.)

Problem 6. **(a)** *Solve $y' + 2y = 2$.*

Solution: When we first did Topic 2, we would have done this using the variation of parameters formula. That would still work, but a better method is the method of optimism: Guess a solution $y = \textit{constant}$. This gives $y_p(t) = 1$. Adding in the general homogeneous solution, the general solution is $\boxed{y(t) = 1 + Ce^{-2t}}$.

(b) *Solve $y' + 2y = 2t$.*

Solution: Again, rather than variation of parameters, we can use undetermined coefficients: Try $y(t) = At + B$. Substitution plus algebra gives $y_p(t) = t - 1/2$. So the general solution is $\boxed{y(t) = y_p(t) + y_h(t) = t - \frac{1}{2} + Ce^{-2t}.}$

(c) *Solve $y' + 2y = 5 + 2t$.*

Solution: Scaling and adding the solutions to $y' + 2y = 2$ and $y' + 2y = 2t$ we get $\boxed{y(t) = \frac{5}{2} + t - \frac{1}{2} + Ce^{-2t} = 2 + t + Ce^{-2t}.}$

Problem 7. **(IVP using definite integrals)**
Solve $xy' - e^x y = 0$, $y(1) = 2$ using definite integrals.

Solution: This is separable: $\dfrac{dy}{y} = \dfrac{e^x\,dx}{x}$. Because we can't compute $\int \frac{e^x}{x}\,dx$ in closed form, we need to give a definite integral solution.

$$\int_2^y \frac{du}{u} = \int_1^x \frac{e^v}{v}\,dv \quad \Rightarrow \log(y) - \log(2) = \int_1^x \frac{e^v}{v}\,dv.$$

Exponentiating we get: $\boxed{y(x) = 2e^{\int_1^x e^v/v\,dv}.}$

Problem 8. *Solve $y' + 2y = 2$; $y(1) = 1$.*

Solution: From an earlier problem we have $y(t) = 1 + Ce^{-2t}$.

The initial condition implies $y(1) = 1 + Ce^{-2} = 1$. So, $C = 0$. Therefore, $\boxed{y(t) = 1}$.

Problem 9. *Show that $y' + y^2 = q$ does not satisfy the superposition principle.*

Solution: We'll do this with a specific counterexample: (It could just as easily be done generally.) Suppose $y_1' + y_1^2 = 1$ and $y_2' + y_2^2 = t$. If superposition were true, then we would have

$$(y_1 + y_2)' + (y_1 + y_2)^2 = 1 + t.$$

But

$$(y_1 + y_2)' + (y_1 + y_2)^2 = y_1' + y_1^2 + y_2' + y_2^2 + 2y_1y_2 = 1 + t + 2y_1y_2 \neq 1 + t.$$

So superposition doesn't hold.

Topic 3. Input response models

Problem 10. *Solve the DE* $x' + 2x = f(t)$, $x(0) = 0$, *where* $f(t) = \begin{cases} 6 & \text{for } 0 \leq t < 1 \\ 0 & \text{for } 1 \leq t < 2 \\ 6 & \text{for } 2 \leq t. \end{cases}$

Solution: First we solve the general cases (you can and should solve these by memory and inpsection).

IVP 1: $x' + 2x = 0$, $x(t_0) = b \quad \Rightarrow x(t) = be^{-2(t-t_0)}$.

IVP 2: $x' + 2x = 6$, $x(t_0) = b \Rightarrow x(t) = 3 + (b-3)e^{-2(t-t_0)}$.

For our problem:
Case $0 \leq t < 1$: DE: $x' + 2x = 6$, $x(0) = 0$.

So, using IVP 2, $x(t) = 3 - 3e^{-2t}$. For the next case: $x_1 = x(1) = 3(1 - e^{-2})$.

Case $1 \leq t < 2$: DE: $x' + 2x = 0$, $x(1) = x_1$.

So, using IVP 1, $x(t) = x_1e^{-2(t-1)}$. For the next case: $x_2 = x(2) = x_1e^{-2} = 3(e^{-2} - e^{-4})$.

Case $2 \leq t$: DE: $x' + 2x = 6$, $x(2) = x_2$.

So, using IVP 2, $x(t) = 3 + (x_2 - 3)e^{-2(t-2)}$.

Putting the cases together:

$$x(t) = \begin{cases} 3(1 - e^{-2t}) & \text{for } 0 \leq t < 1 \\ x_1e^{-2(t-1)} = 3(1 - e^{-2})e^{-2(t-1)} & \text{for } 1 \leq t < 2 \\ 3 + (x_2 - 3)e^{-2(t-2)} = 3 + 3(-1 + e^{-2} - e^{-4})e^{-2(t-2)} & \text{for } 2 \leq t. \end{cases}$$

Topic 4. Complex arithmetic and exponentials

Problem 11. *Polar coordinates: Write* $z = -2 + 3i$ *in polar form.*

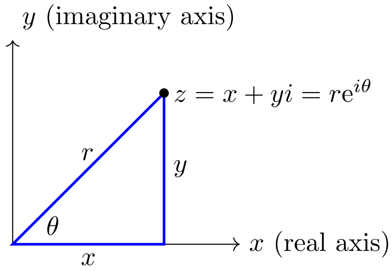
Solution: $|z| = \sqrt{13}$. $\text{Arg}(z) = \theta = \tan^{-1}(-3/2)$ in the 2nd quadrant. $\boxed{z = \sqrt{13}e^{i\theta}.}$

Problem 12. *Write* $3e^{i\pi/6}$ *in rectangular coordinates.*

Solution: By Euler's formula: $3e^{i\pi/6} = 3\cos(\pi/6) + 3i\sin(\pi/6) = \boxed{3\sqrt{3}/2 + i3/2.}$

Problem 13. (Trig triangle)
Draw and label the triangle relating rectangular with polar coordinates.

Solution:



Problem 14. *Compute* $\dfrac{1}{-2+3i}$ *in polar form. Convert the denominator to polar form first. Be sure to describe the polar angle precisely.*

Solution: In polar form $-2+3i = \sqrt{13}e^{i\theta}$, where $\theta = \arg(-2+3i) = \tan^{-1}(-3/2)$ in the second quadrant.

Therefore, $\dfrac{1}{-2+3i} = \dfrac{1}{\sqrt{13}e^{i\theta}} = \dfrac{1}{\sqrt{13}}e^{-i\theta}$.

Problem 15. *Find a formula for* $\cos(3\theta)$ *in terms of* $\cos(\theta)$ *and* $\sin(\theta)$*.*

Solution: First note, $\cos(3\theta) = \text{Re}(e^{3i\theta})$. We know,

$$e^{3i\theta} = (\cos(\theta) + i\sin(\theta))^3 = \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$$

Taking the real part, we have $\boxed{\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta).}$

Problem 16. (Roots)
Find all fifth roots of -2. Give them in polar form. Draw a figure showing the roots in the complex plane.

Solution: We start by writing -2 in polar form, being sure to include all values of the argument:

$$-2 = 2e^{i\pi + i2n\pi}.$$

Raising this to the power $1/5$ gives

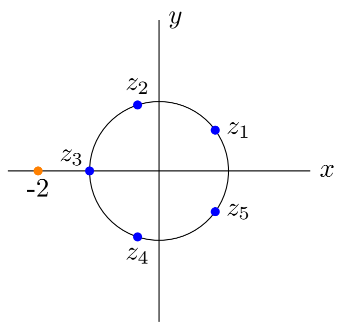
$$(-2)^{1/5} = 2^{1/5}e^{i\pi/5 + i2n\pi/5}.$$

Thus the 5 unique roots are:

$$z_1 = 2^{1/5}e^{i\pi/5}, \quad z_2 = 2^{1/5}e^{i3\pi/5}, \quad z_3 = 2^{1/5}e^{i5\pi/5}, \quad z_4 = 2^{1/5}e^{i7\pi/5}, \quad z_5 = 2^{1/5}e^{i9\pi/5}.$$

The only one of these that simplifies is $z_3 = 2^{1/5}e^{i5\pi/5} = -2^{1/5}$.

The figure below shows -2 and its fifth roots. Notice they are equally spaced around a circle of radius $2^{1/5}$.



Fifth roots of -2

Problem 17. *Compute* $I = \int e^{2x} \cos(3x)\, dx$ *using complex techniques.*

Solution: Complexify: Let $I_c = \int e^{(2+3i)x}$, then $I = \text{Re}(I_c)$.

Integrating: $I_c = \dfrac{e^{(2+3i)x}}{2+3i}$.

In polar form, $2 + 3i = \sqrt{13}e^{i\phi}$, where $\phi = \text{Arg}(2+3i) = \tan^{-1}(3/2)$ in Q1.

So, $I_c = \dfrac{e^{2x}}{\sqrt{13}} e^{i(3x-\phi)}$. Taking the real part: $\boxed{I = \dfrac{e^{2x}}{\sqrt{13}} \cos(3x - \phi)}$.

Problem 18. **(a)** *Show* $\cos(t) = (e^{it} + e^{-it})/2$ *and* $\sin(t) = (e^{it} - e^{-it})/2i$.
(b) *Find all the real-valued functions of the form* $\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it}$.

Solution: **(a)** This follows easily from Euler's formula:

$$\frac{e^{it} + e^{-it}}{2} = \frac{\cos(t) + i\sin(t) + \cos(t) - i\sin(t)}{2} = \cos(t)$$
$$\frac{e^{it} - e^{-it}}{2i} = \frac{\cos(t) + i\sin(t) - (\cos(t) - i\sin(t))}{2i} = \sin(t)$$

(b) Using Euler's formula we know that

$$\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it} = (\tilde{c}_1 + \tilde{c}_2)\cos(t) + i(\tilde{c}_1 - \tilde{c}_2)\sin(t)$$

If this is real-valued, then we must have the coefficients of $\cos(t)$ and $\sin(t)$ are real:
$\tilde{c}_1 + \tilde{c}_2$ real implies $\text{Im}(\tilde{c}_1) = -\text{Im}(\tilde{c}_2)$.
$i(\tilde{c}_1 - \tilde{c}_2)$ real implies $\text{Re}(\tilde{c}_1) = \text{Re}(\tilde{c}_2)$.
Thus $\tilde{c}_1$ and $\tilde{c}_2$ are complex conjugates, say $\tilde{c}_1 = a - ib$ and $\tilde{c}_2 = a + ib$. Then

$$\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it} = 2a\cos(t) + 2b\sin(t)$$

Changing notation slightly, the answer is $\boxed{x(t) = a\cos(t) + b\sin(t).}$

Problem 19. *Find all the real-valued functions of the form $x = \tilde{c}e^{(2+3i)t}$.*

Solution: Let $\tilde{c} = a + ib$. Expanding x we get

$$x(t) = e^{2t}(a+ib)(\cos(3t) + i\sin(3t)) = e^{2t}(a\cos(3t) - b\sin(3t) + i(a\sin(3t) + b\cos(3t)))$$

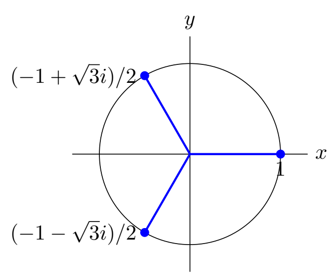
It's clear that the imaginary part can only be 0 if $a = b = 0$. So the only such real-valued function is $x(t) = 0$.

Problem 20. *Find the 3 cube roots of 1 by locating them on the unit circle and using basic trigonometry.*

Solution: We know one cube root is 1. This is on the unit circle and the three roots are evenly spaced around the circle. So the other two are at $e^{2\pi i/3}$ and $e^{4\pi i/3}$. Since $2\pi/3 = 120°$ and $4\pi/3 = 240°$, we can use our knowledge of 30, 60, 90 triangles to write the roots as

$$1, \qquad e^{2\pi i/3} = \frac{-1+\sqrt{3}i}{2}, \qquad e^{4\pi i/3} = \frac{-1-\sqrt{3}i}{2}$$

The figure below shows the three cube roots of 1.



Cube roots of 1

Problem 21. *Express in the form $a + bi$ the 6 sixth roots of 1.*

Solution: In polar form $1 = e^{i2\pi k}$, so

$$\begin{aligned} 1^{1/6} &= e^{i2\pi k/6} = e^{i\cdot 0},\ e^{i\pi/3},\ e^{i2\pi/3},\ e^{i3\pi/3},\ e^{i5\pi/3},\ e^{i5\pi/3} \\ &= 1,\ \frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -1,\ -\frac{1}{2} - i\frac{\sqrt{3}}{2},\ \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{aligned}$$

Problem 22. *Use Euler's formula to derive the trig addition formulas for* sin *and* cos*.*

Solution: Use $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$.

$$\begin{aligned} e^{i\alpha}e^{i\beta} &= (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) \\ &= (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) \\ e^{i(\alpha+\beta)} &= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \end{aligned}$$

Equating the two expressions above, we have:

$$(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

Equating the real and imaginary parts, we get the trig addition formulas:

$$\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \cos(\alpha+\beta)$$
$$\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) = \sin(\alpha+\beta).$$

Topic 5. Constant coefficient linear homogeneous DEs; Damping

Problem 23. **(a)** *Solve $x'' - 8x' + 7x = 0$ using the characteristic equation method.*

Solution: (Model solution) Characteristic equation: $r^2 - 8r + 7 = 0$.

Roots: $r = 7,\ 1$.

General real-valued solution: $x(t) = c_1 e^{7t} + c_2 e^{t}$.

(b) *Solve $x'' + 2x' + 5x = 0$ using the characteristic equation method.*

Solution: Characteristic equation: $r^2 + 2r + 5 = 0$.

Roots: $r = (-2 \pm \sqrt{4-20})/2 = -1 \pm 2i$.

General real-valued solution: $x(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$.

(c) *Assume the polynomial $r^5 + a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0$ has roots*

$$0.5, \quad 1, \quad 1, \quad 2 \pm 3i.$$

Give the general real-valued solution to the homogeneous constant coefficient DE

$$x^{(5)} + a_4 x^{(4)} + a_3 x^{(3)} + a_2 x'' + a_1 x' + a_0 x = 0.$$

Solution: Since we are given the roots, we can write the general solution directly:

$$x(t) = c_1 e^{0.5t} + c_2 e^{t} + c_3 t e^{t} + c_4 e^{2t}\cos(3t) + c_5 e^{2t}\sin(3t).$$

Problem 24. **(Unforced second-order physical systems)**
The DE $x'' + bx' + 4x = 0$ models a damped harmonic oscillator. For each of the values $b = 0, 1, 4, 5$ say whether the system is undamped, underdamped, critically damped or overdamped.

Sketch a graph of the response of each system with initial condition $x(0) = 1$ and $x'(0) = 0$. (It is not necessary to find exact solutions to do the sketch.)

Say whether each system is oscillatory or non-oscillatory.

Solution: The characteristic roots are $\dfrac{-b \pm \sqrt{b^2 - 16}}{2}$. We call the term under the square root the discriminant.

$b = 0$: The system is undamped and oscillatory (in fact sinusoidal).

$b = 1$: The discriminant $= 1 - 16 < 0$, so the roots are complex, which implies the system is underdamped and oscillatory.

$b = 5$: The discriminant is positive, so the roots are real, which implies system overdamped and non-oscillatory.

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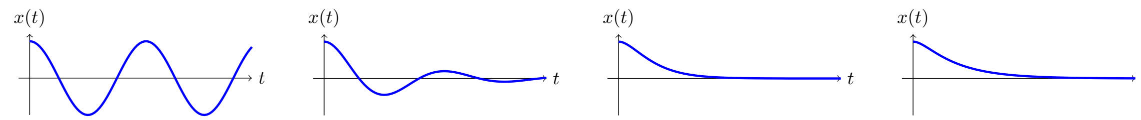
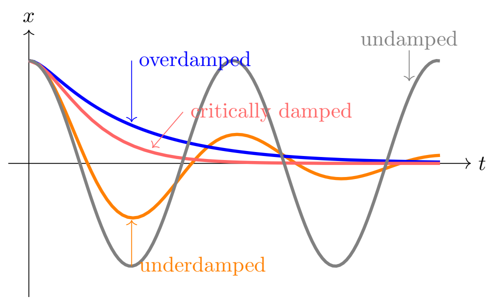
Here are plots of each of these solutions starting from $x(0) = 1$, $x'(0) = 0$.



$b = 0$: undamped $\quad$ $b = 1$: underdamped $\quad$ $b = 4$: crit. damped $\quad$ $b = 5$: overdamped

The following figure is from the Topic 5 notes. It shows the different types of damping, though not necessarily using the coefficients in this problem. Note, that the initial conditions are all the same and, the initial velocity $x'(0) = 0$ causes all the graphs to have a horizontal tangent at $t = 0$.



Problem 25. *State and verify the superposition principle for $mx'' + bx' + kx = 0$, $(m, b, k$ constants).*

Solution: Superposition principle for linear, homogeneous DEs:

If x_1 and x_2 are solutions to the DE then so are all linear combinations $x = c_1x_1 + c_2x_2$.

Proof. Plug x into the DE and then chug through the algebra to show that x is a solution.

$$\begin{aligned} mx'' + bx' + kx &= m(c_1x_1 + c_2x_2)'' + b(c_1x_1 + c_2x_2)' + k(c_1x_1 + c_2x_2) \\ &= c_1mx_1'' + c_2mx_2'' + c_1bx_1' + c_2bx_2' + c_1kx_1 + c_2kx_2 \\ &= c_1 \underbrace{mx_1'' + bx_1' + kx_1}_{\substack{\text{0 by assumption that} \\ x_1 \text{ is a solution}}} + c_2 \underbrace{mx_2'' + bx_2' + kx_2}_{\substack{\text{0 by assumption that} \\ x_2 \text{ is a solution}}} \\ &= 0 \qquad \blacksquare \end{aligned}$$

Problem 26. *A constant coefficient, linear, homogeneous DE has characteristic roots*

$$-1 \pm 2i,\ -2,\ -2,\ -3 \pm 4i.$$

(a) *What is the order of the DE? (Notice the $\pm$ in the list of roots.)*

Solution: 6 roots implies it is a 6th order DE.

(b) *What is the general, real-valued solution.*

Solution: The 6 roots give 6 basic solutions:

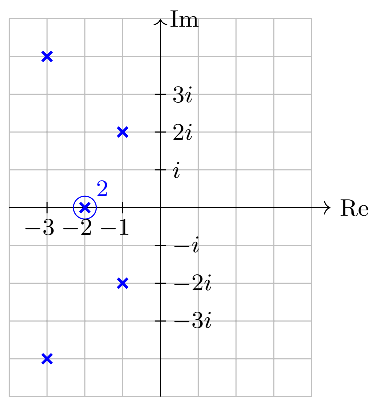
$$x_1 = e^{-t}\cos(2t) \qquad x_2 = e^{-t}\sin(2t)$$
$$x_3 = e^{-2t} \qquad x_4 = te^{-2t}$$
$$x_5 = e^{-3t}\cos(4t) \qquad x_6 = e^{-3t}\sin(4t)$$

The general solution is

$$x(t) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6.$$

(c) *Draw the pole diagram for this system. Explain why it shows that all solutions decay exponentially to 0. What is the exponential decay rate of the general solution?*

Solution: For the pole diagram, we put an x at each root. We indicate the double root by circling it and putting a small 2 as a superscript.



Since all the poles are in the left half plane, all the basic solutions have negative exponents, i.e., decay exponentially to 0. This implies that all solutions, which are linear combinations of the basic ones, decay exponentially.

The decay rate is controlled by the right-most root. In this case, this has real part -1, so the general solution decays like e^{-t}.

Topic 6. Exponential Response Formula

Problem 27. *Let $P(D) = D^2 + 8D + 7$. Find the general real-valued solution to the following.*

For oscillatory answers your particular solutions should be in amplitude-phase form.

(a) $P(D)x = e^{2t}$.

Solution: For all parts $x_h(t) = c_1e^{-t} + c_2e^{-7t}$ and the general solution is $x = x_p + x_h$.

Using the exponential response formula (ERF): $x_p(t) = \dfrac{e^{2t}}{P(2)} = \dfrac{e^{2t}}{27}$.

(b) $P(D)x = \cos(3t)$.

Solution: Using the sinusoidal response formula (SRF): $x_p(t) = \dfrac{\cos(3t-\phi)}{|P(3i)|}$, where $\phi = \text{Arg}(P(3i))$.

$P(3i) = -2 + 24i$: $|P(3i)| = \sqrt{580}$, $\phi = \text{Arg}(P(3i)) = \tan^{-1}(-12)$ in Q2.

(c) $P(D)x = e^{2t}\cos(3t)$.

Solution: First we complexify: $z'' + 8z' + 7z = e^{(2+3i)t}$, $x = \text{Re}(z)$.

The ERF gives

$$z_p(t) = \frac{e^{(2+3i)t}}{P(2+3i)} = \frac{e^{2t}e^{i(3t-\phi)}}{|P(2+3i)|},$$

where $\phi = \text{Arg}(P(2+3i))$.

Computing: $P(2+3i) = 18+36i$: $\boxed{|P(3i)| = 18\sqrt{5}}$, $\boxed{\phi = \text{Arg}(P(2+3i)) = \tan^{-1}(2) \text{ in Q1.}}$

So, $x_p(t) = \text{Re}(z_p(t)) = \dfrac{e^{2t}\cos(3t-\phi)}{|P(2+3i)|}$.

(d) $P(D)x = e^{-t}$.

Solution: Since $P(-1) = 0$, we need to use the extended ERF:

$$x_p(t) = \frac{te^{-t}}{P'(-1)} = \boxed{\frac{te^{-t}}{6}.}$$

Topic 7. Undetermined coefficients; Theory

Problem 28. *Find the general solution to* $x' + 3x = t^2 + 3$

Solution: Try $x = At^2 + Bt + C$. Substituting we get $(2At + B) + 3(At^2 + Bt + C) = t^2 + 3$. Equating coefficients gives

$$\begin{aligned} t^2: &\quad 3A = 1 \\ t: &\quad 2A + 3B = 0 \\ 1: &\quad B + 3C = 3. \end{aligned}$$

Solving the algebraic system gives $A = 1/3$, $B = -2/9$, $C = 29/27$. So, $x_p(t) = \frac{t^2}{3} - \frac{2}{9}t + \frac{29}{27}$, The general solution (including the homogeneous piece) is

$$x(t) = x_p(t) + x_h(t) = \frac{t^2}{3} - \frac{2}{9}t + \frac{29}{27} + ce^{-3t}$$

Problem 29. *Find one solution to* $x''' + 3x'' + 2x' + 5x = 4$.

Solution: Try a constant solution: $\boxed{x(t) = 4/5.}$

Problem 30. *Find the general solution to* $x'' + 3x' = t + 1$.

Solution: Because the DE only goes as far as x' we have to bump the powers of our optimistic guess up by 1: Try $x = At^2 + Bt$. Substituting gives

$$2A + 3(2At + B) \ = t + 1.$$

Equating coefficients gives

$$\begin{aligned} t: & \quad 6A = 1 \\ 1: & \quad 2A + 3B = 1. \end{aligned}$$

We get, $A = 1/6$, $B = 2/9$, so $x_p(t) = \frac{1}{6}t^2 + \frac{2}{9}t$.

Homogeneous solution: characteristic equation $r^2 + 3r = 0 \quad \Rightarrow$ roots $= -3,\ 0$.

So, $x_h(t) = c_1 e^{-3t} + c_2$ and the general solution is

$$x(t) = x_p(t) + x_h(t) = \frac{1}{6}t^2 + \frac{2}{9}t + c_1 e^{-3t} + c_2$$

Topic 8. Stability

Stability is about the system not the input.

Problem 31. *Is the system $x'' + x' + 4x = 0$ stable?*

Solution: Short answer: second-order with positive coefficients implies $\boxed{\textit{stable}}$.

Longer answer: characteristic roots are $r = \frac{-1 \pm \sqrt{1-16}}{2}$. Since both roots have a negative real part, the system is stable.

Problem 32. *Is a 4th order system with roots $\pm 1, -2 \pm 3i$ stable. Which solutions to the homogeneous DE go to 0 as $t \to \infty$?*

Solution: No, the root $r = 1$ is positive so the system is not stable.

The general homogeneous solution is

$$x_h(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t}\cos(3t) + c_4 e^{-2t}\sin(3t),$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

The solutions that go to 0 are the ones with $c_1 = 0$, i.e., those of the form

$$x(t) = c_2 e^{-t} + c_3 e^{-2t}\cos(3t) + c_4 e^{-2t}\sin(3t),$$

where c_2, c_3, c_4 are arbitrary constants.

Problem 33. *For what k is the system $x' + kx = 0$ stable?*

Solution: Since the characteristic root is $r = -k$, this is stable when $k > 0$.

A better way to see this is, if $k > 0$ the system is one of exponential decay. If $k < 0$ it is one of exponential growth. If $k = 0$ it is an edge case. Some people will say it's stable but not asymptotically stable.

Topic 9. Amplitude response, resonance and practical resonance

Problem 34. *Consider the system* $x'' + 8x = F_0 \cos(\omega t)$*.*

(a) *Why is this called a driven undamped system?*

Solution: This models an undamped harmonic oscillator (with mass = 1). The input $F_0 \cos(\omega t)$ is a force which 'drives' the motion of system.

Note: we assumed that $F_0 \cos(\omega t)$ is the input. Not matter what we call it, the right side of the equation represents a force that drives the system.

(b) *Solve this using the sinusoidal response formula (SRF). Then do it again using complex replacement and the exponential response formula (ERF).*

Solution: Using the SRF and Extended SRF we have

For $\omega \neq \sqrt{8}$: $P(i\omega) = 8 - \omega^2 = |8 - \omega^2| e^{i\phi(\omega)}$, where $\phi(\omega) = \begin{cases} 0 & \text{if } 8 - \omega^2 > 0 \\ \pi & \text{if } 8 - \omega^2 < 0. \end{cases}$

So, $x_p(t) = \dfrac{F_0 \cos(\omega t - \phi(\omega))}{|8 - \omega^2|} = \begin{cases} F_0 \cos(\omega t) & \text{if } \omega < \sqrt{8} \\ F_0 \cos(\omega t - \pi) = -F_0 \cos(\omega t) & \text{if } \omega > \sqrt{8}. \end{cases}$

If $\omega = \sqrt{8}$, we need the Extended SRF: $x_p(t) = \dfrac{F_0 t \cos(\omega t - \phi)}{|P'(i\omega)|}$, where $\phi = \text{Arg}(P'(i\omega))$.

$P'(r) = 2r$, so $P'(i\sqrt{8}) = 2i\sqrt{8} = 2\sqrt{8} e^{i\pi/2}$. Therefore,

$$x_p(t) = \frac{F_0 t \cos(\sqrt{8}\, t - \pi/2)}{2\sqrt{8}}.$$

Alternatively, we could use complex replacement to arrive at the same formula.

Complexify: $z'' + 8z = F_0 e^{i\omega t}$, $x = \text{Re}(z)$.

Characteristic polynomial: $P(r) = r^2 + 8$.

Exponential response formula: As above, $P(i\omega) = |P(i\omega)| e^{i\phi}$, where

$$|P(i\omega)| = |8 - \omega^2| \quad \text{and } \phi(\omega) = \text{Arg}(P(i\omega)) = \begin{cases} 0 & \text{if } 8 - \omega^2 > 0 \\ \pi & \text{if } 8 - \omega^2 < 0. \end{cases}$$

So, if $\omega \neq \sqrt{8}$, then

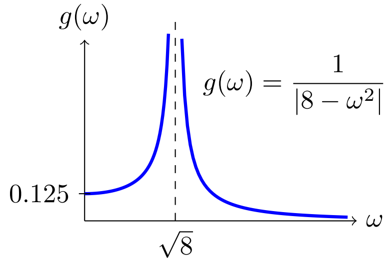
$$z_p(t) = \frac{F_0 e^{i\omega t}}{P(i\omega)} = \frac{F_0 e^{i(\omega t - \phi(\omega))}}{|8 - \omega^2|}, \text{ so } x_p(t) = \text{Re}(z(t)) = \frac{F_0 \cos(\omega t - \phi(\omega))}{|8 - \omega^2|}$$

If $\omega = \sqrt{8}$, then $P(i\omega) = 0$, and we need to use the extended exponential response formula. As usual, we label the natural frequency $\omega_0 = \sqrt{8}$.

$$z_p(t) = \frac{F_0 t e^{i\omega_0 t}}{P'(i\omega_0)} = \frac{F_0 t e^{i\omega_0 t}}{2i\omega_0} = \frac{F_0 t e^{i\omega_0 t}}{2\omega_0 e^{i\pi/2}} = \frac{F_0 t e^{i(\omega_0 t - \pi/2)}}{2\omega_0} \quad \text{so, } x(t) = \text{Re}(z(t)) = \frac{F_0 t \cos(\omega_0 t - \pi/2)}{2\omega_0}.$$

(c) *Consider the right hand side of the DE to be the input. Graph the amplitude response function.*

Solution: $g(\omega) = 1/|8-\omega^2|$. The graph of g vs. ω has a vertical asymptote at $\omega = \sqrt{8}$.



(d) *What is the resonant frequency of the system?*

Solution: Resonant frequency at $\omega = \sqrt{8}$.

(e) *Why is this called the natural frequency?*

Solution: Because the unforced system $x'' + 8x = 0$ will oscillate at this frequency

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Problem 35. *Consider the forced damped system: $x'' + 2x' + 9x = \cos(\omega t)$.*

(a) *What is the natural frequency of the system?*

Solution: Natural frequency = frequency of unforced, undamped system = $\sqrt{9} = 3$.

(b) *Find the response of the system in amplitude-phase form.*

Solution: We can go right to the sinusoidal response formula: $\boxed{x_p(t) = \dfrac{1}{|P(i\omega)|}\cos(\omega t - \phi(\omega))}$,

where, $P(r) = r^2 + 2r + 9 \Rightarrow P(i\omega) = 9 - \omega^2 + 2i\omega$.

So, $\boxed{|P(i\omega)| = \sqrt{(9-\omega^2)^2 + 4\omega^2}}$, and $\boxed{\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1}\dfrac{2\omega}{9-\omega^2} \text{ in Q1 or Q2.}}$

Alternatively (and you should know how to do this)

Complexify: $z'' + 2z' + 9z = e^{i\omega t}$, $x = \text{Re}(z)$.

Char. polynomial: $P(r) = r^2 + 2r + 9$, so $P(i\omega) = 9 - \omega^2 + 2i\omega$.

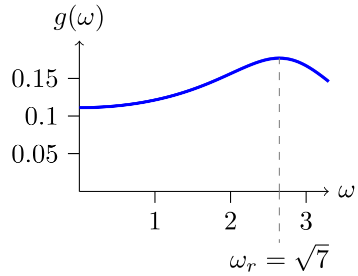
ERF: $z_p(t) = \dfrac{e^{i\omega t}}{P(i\omega)} = \dfrac{e^{i\omega t}}{9-\omega^2+2i\omega}$

Polar form: $9-\omega^2+2i\omega = \sqrt{(9-\omega^2)^2+4\omega^2}\,e^{i\phi(\omega)}$, where $\boxed{\phi(\omega) = \tan^{-1}(2\omega/(9-\omega^2)), \text{ in Q1 or Q2.}}$

Thus, $z_p(t) = \dfrac{1}{\sqrt{(9-\omega^2)^2+4\omega^2}}e^{i(\omega t - \phi(\omega))}$. This implies $\boxed{x_p(t) = \dfrac{1}{\sqrt{(9-\omega^2)^2+4\omega^2}}\cos(\omega t - \phi(\omega)).}$

(c) *Consider the right hand side of the DE to be the input. What is the amplitude response of the system? Draw its graph –be sure to label your axes correctly*

Solution: Amplitude response = gain = $g(\omega) = \dfrac{1}{\sqrt{(9-\omega^2)^2+4\omega^2}}$.



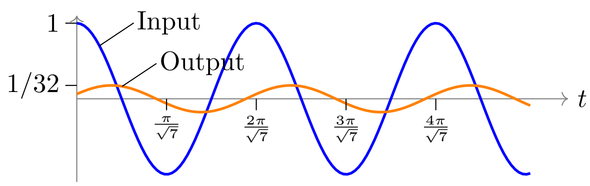
Gain curve and practical resonant frequency

(d) *What is the practical resonant frequency?*

Solution: Practical resonance is where $g(\omega)$ has a maximum. This is the same as where the expression under the radical, $(9 - \omega^2)^2 + 4\omega^2$, has a minimum. Simple calculus shows this is at $\omega_r = \sqrt{7}$.

(e) *When $\omega = \sqrt{7}$ by how many radians does the output peak lag behind the input peak?*

Solution: If $\omega = \sqrt{7}$ then $\phi = \tan^{-1}\sqrt{7} = 1.2$ radians and $g(\sqrt{7}) = 1/\sqrt{32} \approx 0.177$.



(f) *For the forced undamped system $x'' + 9x = \cos(\omega t)$ give a detailed description of the phase lag for different input frequencies?*

The SRF gives the sinusoidal solution: $x_p(t) = \dfrac{\cos(\omega t - \phi(\omega))}{|P(i\omega)|}$, where $\phi(\omega) = \text{Arg}(P(i\omega))$.
So the phase lag at ω is given by $\phi(\omega) = \text{Arg}(P(i\omega))$. We have

$$P(i\omega) = 9 - \omega^2 \;\Rightarrow\; \boxed{\phi(\omega) = \text{Arg}(P(i\omega)) = \begin{cases} 0 & \text{if } \omega < 3 \\ \pi & \text{if } \omega > 3. \end{cases}}$$

Since there is no sinusoidal response when $\omega = 3$ there is no official phase lag, but the solution $x_p(t) = t\cos(3t - \pi/2)/6$ suggests that $\pi/2$ might be a reasonable unofficial choice.

Problem 36. *Consider the driven first-order system: $x' + kx = kF_0\cos(\omega t)$. We'll take the input to be $F_0\cos(\omega t)$. Solve the DE. Find the amplitude response. Show there is never practical resonance.*

Solution: Use the sinusoidal response formula: $x_p(t) = \dfrac{kF_0\cos(\omega t - \phi(\omega))}{|P(i\omega)|}$.

Characteristic polynomial: $P(r) = r + k$. So, $P(i\omega) = i\omega + k$.

Polar form: $P(i\omega) = k + i\omega \Rightarrow \boxed{|P(i\omega)| = \sqrt{k^2 + \omega^2}}$, and $\boxed{\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1}(\omega/k), \text{ in Q1.}}$
So,

$$\boxed{x_p(t) = \frac{kF_0}{\sqrt{k^2 + \omega^2}}\cos(\omega t - \phi(\omega))}$$

Amplitude response = gain = $g(\omega) = \dfrac{k}{\sqrt{k^2+\omega^2}}$. This is a decreasing function in ω, so there is no positive local maximum, i.e., no practical resonance.

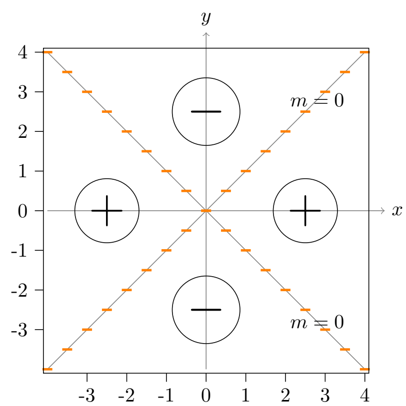
Topic 10. Direction fields, integral curves
Topic 11. Numerical methods: Euler's method

Problem 37. *Consider $y' = x^2 - y^2$*

(a) *Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as + and negative slope as −. Use this to give a very rough sketch of some solution curves.*

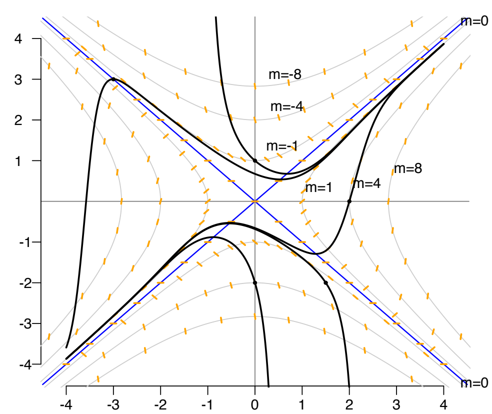
Note: the nullcline consists of two lines.

Solution: The nullcline consists of the lines $y = \pm x$. Below is a sketch of the nullcline with the regions marked + or −. Look at the figure with Part (b) for some integral curves.



(b) *Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.*

Solution: Isoclines are hyperbolas with asymptotes $y = \pm x$.



(c) *Add some integral curves to the plot in Part (b). Include the one with $y(2) = 0$.*

Solution: See plot in Part (b).

(d) *Use squeezing to estimate $y(100)$ for the solution with IC $y(2) = 0$.*

Solution: We can see from the plot in Part (b) that this solution seems to go asymptotically to the nullcline $y = x$.

The argument to see this is a little subtle. We'll give the argument as a sequence of observations. On an exam, you could just state this as an empirical observation about the isoclines sketch.

1. Clearly the nullcline $y = x$ is an upper fence for this integral curve, so the curve stays below this line.

2. To be specific, let's take $m = 2$. The isocline for $m = 2$ goes asymptotically to the line $y = x$. That is, its slope as a curve (not the isocline slope) is close to 1 for large x. Thus, when x is large, the isocline $m = 2$ is a lower fence, i.e., its slope element goes from below to above the isocline.

3. Let x be large enough that the isocline for $m = 2$ is a lower fence. If the integral curve $y(x)$ is below the isocline then its slope is bigger than 2. This means it is growing faster than the isocline and must eventually cross it. At this point it is above the fence and in the funnel between $y = x$ and the isocline for $m = 2$.

4. This funnel goes asymptotically to $y = x$, so we can estimate $y(100) \approx 100$.

(e) *Use Euler's method with $h = 0.5$ to estimate $y(3)$ for the solution with $y(2) = 0$.*

Solution: As in the Topic 11 notes, set up a table with columns: n, x_n, y_n, m, mh.

n	x_n	y_n	m	mh
0	2	0	4	2
1	2.5	2	2.25	1.125
2	3	3.125		

(f) *Is the estimate in Part (e) too high or too low?*

Solution: We can take the derivative of our equation to get the equation for the second derivative $y'' = 2x - 2yy'$. If we look at the point $(x, y) = (2, 0)$, then we can use our original equation to get $y' = 4$, and the second derivative equation to get $y'' = 4 > 0$. A positive second derivative implies the integral curve is concave up, which implies that our estimate is an underestimate, since drawing tangent lines to the curve produces values above the curve.

Topic 12. Autonomous first-order DEs

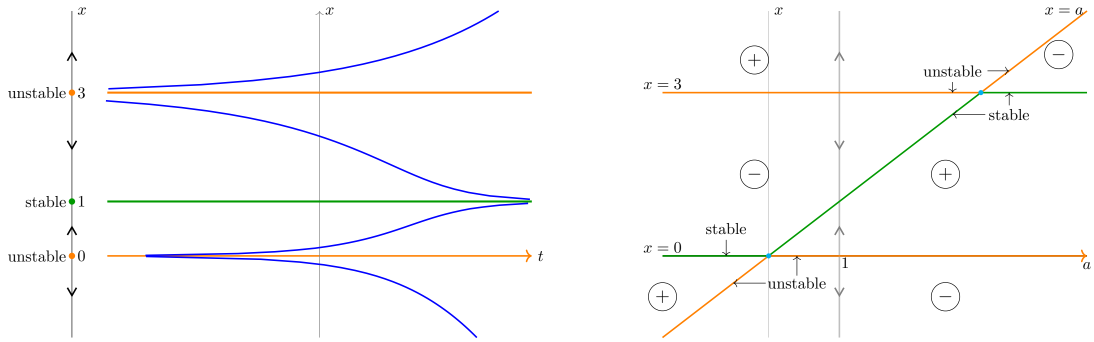
Problem 38. *Let $x' = x(x - a)(x - 3)$*

(a) *Let $a = 1$ draw phase line. Identify type of each critical point, sketch solution graphs.*

Solution: Critical points: $x' = x(x-1)(x-3) = 0 \Rightarrow x = 0, 1, 3$. It is easy to check the sign in each region of the phase line. See the figure below for the phaseline and solution graphs.

(b) *Considering a to be a parameter: draw the bifurcation diagram: identify the stable and unstable branches.*

Solution: The critical points are $x = 0, a, 3$. These lines are plotted in the ax-plane and divide the plane into regions. The phase line from Part (a) was added. It goes through 4 of the 6 regions and allows us to see whether x' is positive or negative in those 4 regions. A simple calculation allowed us to test the remaining two regions, which were then labeled with a plus or minus. The stable critical points are labeled and plotted in green, the unstable are in orange and the semistable in cyan.



Left: (Part (a)) Phase line, sketch of integral curves. Right: (Part (b)) Bifurcation diagram.

(c) *If this models a population, for what a is the population sustainable?*

Solution: The population is sustainable whenever there is a positive stable critical point, i.e., for all $a > 0$.

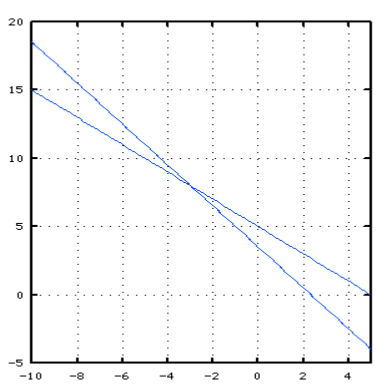
Topic 13. Linear algebra: linearity, vector spaces, connection to DEs
Topic 14. Linear algebra: row reduction, column space, pivots

Problem 39. *Solve this system of linear equations. How many methods can you think of to solve this system?*

$$\begin{aligned} x + y &= 5 \\ 3x + 2y &= 7 \end{aligned}$$

Solution: Some ideas:
(1) Graphically with interesecting lines.
(2) Elimination.
(3) Row reduce the augmented matrix.
(4) Matrix inverse.

(1) $y = -x + 5$ and $y = \frac{7}{2} - \frac{3}{2}x$ are two straight lines of different slopes; so they meet at a single point. To find where, we could eyeball the picture—maybe $(-3, 8)$? That satisfies both equations!



(2) We can use elimination: Subtract 3 times the first equation from the second. Retaining the first equation as well, we get

$$\begin{aligned} x + y &= 5 \\ 0 - y &= -8 \end{aligned}$$

and then the first equation gives $x = -3$. In fact, as a second step, we could add the new second equation to the first one:

$$\begin{aligned} x + 0 &= -3 \\ 0 - y &= -8 \end{aligned}$$

Thus $(x, y) = (-3, 8)$ is the solution.

(3) Matrix methods: The system is $\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$. So,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = -\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$$

Again $(x, y) = (-3, 8)$ is the solution.

Problem 40. *Let* $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. *Suppose R is the row reduced echelon form for A.*

(a) *What is the rank of A?*

Solution: A and R have the same rank. Two pivots in R implies rank $= 2$.

(b) *Find a basis for the null space of A.*

Solution: A and R have the same null space. The second and fourth variables are free. Setting them to 1 and 0 in turn gives a basis. As usual, we organize the computation in rows below the matrix:

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & -1 & 1 \end{matrix}$$

So the basis consists of the vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

(c) *Suppose the column space of A has basis* $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. *Find a possible matrix for A. That is, give a matrix A with RREF R and the given column space.*

Solution: Looking at R the Columns 1 and 3 are pivot columns. We put the given basis in those columns:

$$A = \begin{bmatrix} 1 & * & 3 & * \\ 1 & * & 1 & * \\ 0 & * & 1 & * \end{bmatrix}$$

The free columns of R are linear combinations of the pivot columns and those of A are the same linear combinations. In R it is clear that

$$\text{Col}_2 = 2 \times \text{Col}_1 \text{ and } \text{Col}_4 = 3 \times \text{Col}_1 + \text{Col}_3.$$

So,

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(d) *Find a matrix with the same row reduced echelon form, but such that* $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *are in its column space.*

Solution: We found the relationships between the columns in Part (c). So we put the given columns as pivot columns and construct the free columns from these relationships:
$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 3 & 6 \end{bmatrix}$$

Note: you could put any other basis for the subspace generated by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the pivot columns and adjust the free columns accordingly.

Problem 41. *Consider the following system of equations:*

$$\begin{aligned} x + y + z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned}$$

(a) *Write this system of equations as a matrix equation.*

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$$

(b) *Use row reduction to get to row echelon form. What is the solution set?*

Solution: Set up the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 7 \\ 1 & 3 & 6 & 11 \end{array}\right]$$

Do row reduction to RREF

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 7 \\ 1 & 3 & 6 & 11 \end{array}\right] \xrightarrow[\text{Row}_3 = \text{Row}_3 - \text{Row}_1]{\text{Row}_2 = \text{Row}_2 - \text{Row}_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 5 & 6 \end{array}\right] \xrightarrow{\text{Row}_3 = \text{Row}_3 - 2\text{Row}_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[\text{Row}_1 = \text{Row}_1 - \text{Row}_3]{\text{Row}_2 = \text{Row}_2 - 2\text{Row}_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{\text{Row}_1 = \text{Row}_1 - \text{Row}_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The solution is $x = 5$, $y = -2$, $z = 2$. You can check this by substituting it into the original equations.

Problem 42. *Solve the following equation using row reduction:*

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) *At the end of the row-reduction process, was the last column pivotal or free? Is this related to the absence of solutions?*

Solution: The augmented matrix is $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 6 & 0 \end{array}\right]$.

Do row reduction:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{\text{Row}_2 = \text{Row}_2 - 3\text{Row}_3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -3 \end{array}\right] \xrightarrow{\text{Row}_2 = -\text{Row}_2/3} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

The last equation now reads $0x + 0y = 1$, which is rather hard to satisfy.

(We could already see this problem after the first reduction step.)

The last column was pivotal. This implies there is a row in the augmented RREF matrix with all zeros except for a 1 in the last column. This row corresponds to the equation $0x + 0y = 1$, which explains why there are no solutions.

(b) *Find a new vector* $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ *such that* $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ *has a solution.*

Solution: Well, we could always take $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, because the equation is then obviously solved by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

To be more general, we can take $\mathbf{b}$ in the column space of the coefficient matrix. The row reduced echelon form shows that Column 1 is the only pivot column. So the column space has basis $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Thus, the vectors $c \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are exactly the vectors for which the equation admits a solution.

Problem 43. *Consider the matrix* $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & 6 & 2 & 4 \\ 0 & 0 & 10 & 3 & 6 \end{bmatrix}$. *Put A in row reduced echelon form. Find the column space, null space, rank, a basis for the column space, a basis for the null space, the dimension of each of the spaces.*

Solution: Here is the row reduction to RREF:

$$A \xrightarrow{\text{Row}_2 = \text{Row}_2 - 2\text{Row}_1} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 3 & 6 \end{bmatrix} \xrightarrow{\text{Swap Row}_2 \text{ and Row}_3} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 10 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{Scale Row}_2 \text{ by } 1/10} \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & 3/10 & 6/10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row}_1 = \text{Row}_1 - 3\text{Row}_2} \begin{bmatrix} 1 & 2 & 0 & 1/10 & 2/10 \\ 0 & 0 & 1 & 3/10 & 6/10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot Columns 1 and 3 of A give a basis for Col(A).

$$\text{Col}(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} \right\}$$

Rank(A) = # of pivots = dimension of Col(A) = 2.

Null(A) has dimension 3 = # of free variables.

A basis for Null(A) is found by setting the free variables x_2, x_4, x_5 alternately to 1 and 0. Using our usual format we get:

$$\begin{bmatrix} 1 & 2 & 0 & 1/10 & 2/10 \\ 0 & 0 & 1 & 3/10 & 6/10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ -2 & 1 & 0 & 0 & 0 \\ -1/10 & 0 & -3/10 & 1 & 0 \\ -2/10 & 0 & -6/10 & 0 & 1 \end{array}$$

Basis of Null(A) is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/10 \\ 0 \\ -3/10 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/10 \\ 0 \\ -6/10 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$\text{Null}(A) = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1/10 \\ 0 \\ -3/10 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2/10 \\ 0 \\ -6/10 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Problem 44. **(a)** *Suppose we have a matrix equation*

$$\begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & x \end{bmatrix}$$

Can you specify x? For any value of x you think is allowable, find such an equation. Can any of the $\bullet$'s be 0?

Solution: Each column of the product is a multiple of the column vector in the first factor. The 1s show that they are the same multiple. So x must be 2.

Alternatively, each row of the product is a multiple of the row vector. The first column of the product shows that the second row must be twice the first row. So x must be 2.

One equation that works for $x = 2$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

None of the $\bullet$'s can be 0, since that would make the corresponding row or column in the product $\mathbf{0}$.

(b) *Suppose we have a matrix equation*

$$\begin{bmatrix} \bullet & 3 \\ \bullet & 4 \\ \bullet & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Can you specify the $\bullet$'s?

Solution: The matrix equation says: the first column plus twice the second column is zero. So the first column must be $\begin{bmatrix} -6 \\ -8 \\ -10 \end{bmatrix}$.

(c) *Suppose we have a matrix equation*

$$\begin{bmatrix} x & 3 \\ y & 4 \\ z & 5 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$?

Solution: The equation says that the columns of the matrix form a linearly dependent set. That is: one is a multiple of the other. Since the second column is nonzero, we can be sure that the first is a multiple of the second: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ for some t.

Problem 45. *Suppose we have a matrix equation*

$$\begin{bmatrix} 1 & x & 2 \\ 3 & y & 4 \\ 5 & z & 6 \end{bmatrix} \mathbf{c} = \mathbf{0}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$?

Solution: To have a nontrivial null space the rank must be less than 3. Since the first and third columns are independent, the middle column must be a linear combination of them.

Geometrically, the middle column is in the plane containing the origin and the other two columns.

Problem 46. *For what values of y is it the case that the columns of* $\begin{bmatrix}1&1&2\\3&y&4\\5&1&6\end{bmatrix}$ *form a linearly independent set?*

Solution: The columns are linearly independent when the matrix has rank 3. We can find the rank by row reduction:

$$\begin{bmatrix}1&1&2\\3&y&4\\5&1&6\end{bmatrix}\xrightarrow[R_3=R_3-5R_1]{R_2=R_2-3R_1}\begin{bmatrix}1&1&2\\0&y-3&-2\\0&-4&-4\end{bmatrix}\xrightarrow{\text{swap } R_2 \text{ and } R_3}\begin{bmatrix}1&1&2\\0&-4&-4\\0&y-3&-2\end{bmatrix}$$

$$\xrightarrow{R_2=-R_2/4}\begin{bmatrix}1&1&2\\0&1&1\\0&y-3&-2\end{bmatrix}\xrightarrow{R_3=R_3-(y-3)R_2}\begin{bmatrix}1&1&2\\0&1&1\\0&0&1-y\end{bmatrix}$$

If $1-y\neq 0$, then we have 3 pivots. So the columns are linearly independent exactly when $y\neq 1$.

Problem 47. *For the matrix* $A=\begin{bmatrix}1&2&3\\2&3&4\\3&4&5\end{bmatrix}$*:*

(a) *Find the row reduced echelon form of A; call it R.*

Solution: Here are the row reduction steps:

$$A=\begin{bmatrix}1&2&3\\2&3&4\\3&4&5\end{bmatrix}\xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1}\begin{bmatrix}1&2&3\\0&-1&-2\\0&-3&-6\end{bmatrix}\xrightarrow[R_3=R_3+3R_2]{R_2=-R_2}\begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}=R$$

(b) *The last column of R should be a linear combination of the first columns in an obvious way. This is a linear relation among the columns of R. Find a vector $\mathbf{x}$, such that $R\mathbf{x}=\mathbf{0}$, which expresses this linear relationship.*

Solution: This is just an awkward way of asking about null vectors. The first two columns are pivotal and the third is free. So the third is a combination of the other two. By inspection of R, we see that

$$\text{Col}_3=-\text{Col}_1+2\text{Col}_2 \quad\Rightarrow\quad \text{Col}_1-2\text{Col}_2+\text{Col}_3=\mathbf{0}.$$

As a matrix equation this is $R\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$

(c) *Verify that the same relationship holds among the columns of A.*

Solution: The third column is indeed minus the first plus twice the second. As a matrix equation,

$$A \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(d) *Explain why the linear relations among the columns af R are the same as the linear relations among the columns of A. In fact, explain why, if A and B are related by row transformations, the linear relations among the columns of A are the same as the linear relations among the columns of B.*

Solution: Row transformations do the same thing to the entries of all columns.

This is the same as saying that if A and B are related by row-operations, then their null spaces coincide. Their column spaces usually do not.

Problem 48. *This continues the previous problem. Now, suppose we want to solve $A\mathbf{x} = \mathbf{b}$, where, again, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.*

(a) *When is this possible? Answer this in the form: "$\mathbf{b}$ must be a linear combination of the two vectors ..."*

Solution: The equation can be solved exactly when $\mathbf{b}$ is in the column space of A. In the previous problem, we discovered by row reduction that Column 3 is free and the other two are pivotal. So the first two form a basis of the column space. Therefore, we can say: $\mathbf{b}$ must be a linear combination of the first two columns of A, i.e., $\mathbf{b} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Note. Since there are lots of other possible bases for the column space, this is just one of many possible answers.

(b) *$A\mathbf{x} = \mathbf{b}$ is certainly solvable for $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. (What is the obvious particular solution?) Describe the general solution to this equation, as $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$.*

Solution: $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the first column of A, so the obvious solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We can take this (or any other solution!) as $\mathbf{x_p}$. To get the general solution, we must add the general homogeneous solution.

In the previous problem, we saw there is one free variable and we found a basis vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

for Null(A). Thus, the homogeneous solution is $\mathbf{x_h} = c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. We have

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+c \\ -2c \\ c \end{bmatrix}.$$

Problem 49. *Suppose that the row reduced echelon form of the 4×6 matrix B is*

$$R = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) *Find a linearly independent set of vectors of which every vector in the null space of B is a linear combination.*

Solution: This is just another way of asking for a basis of Null(B). The null space of B is the same as the null space of its row-echelon form.

The free variables are: x_1, x_3 and x_6. As usual, we set them to 1 in turn.

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & -7 & -9 & 1 \end{array}$$

Thus a basis of Null(B) is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ -7 \\ -9 \\ 1 \end{bmatrix} \right\}$$

(b) *Write the columns of B as $\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_6}$. What is $\mathbf{b_1}$? What can we say about $\mathbf{b_2}$? Which of these vectors are linearly independent of the preceding ones? Express the ones which are not independent as explicit linear combinations of the previous ones. Describe a linearly independent set of vectors of which every vector in the column space of B is a linear combination.*

Solution: $\mathbf{b_1}$ must be $\mathbf{0}$, since applying row operations to it give $\mathbf{0}$, and row operations are reversible.

$\mathbf{b_2} \neq \mathbf{0}$. This is all we can say.

The linear relations among the columns of B are the linear relations among the columns of R: so the columns of B corresponding to the pivot columns of R are independent of the previous columns: $\mathbf{b_2}$, $\mathbf{b_4}$, and $\mathbf{b_5}$.

For the linear relations, we just copy what we know for R: $\mathbf{b_1} = 0$; $\mathbf{b_3} = 3\mathbf{b_2}$; $\mathbf{b_6} = 5\mathbf{b_2} + 7\mathbf{b_4} + 9\mathbf{b_5}$.

By a 'linearly independent set of vectors of which every vector in the column space is a linear combination' we just mean a basis of Col(B), These are the pivot columns. That is, $\{\mathbf{b_2}, \mathbf{b_4}, \mathbf{b_5}\}$ is a basis for the column space of B.

Topic 15. Linear algebra: transpose, inverse, determinant

Problem 50. *Compute the transpose of the following matrices.*

(a) $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 2 & 4 \\ 8 & 16 & 32 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 2 \\ 4 & 8 \\ 16 & 32 \end{bmatrix}$

Solution: $A^T = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ $B^T = \begin{bmatrix} 1 & 8 \\ 2 & 16 \\ 4 & 32 \end{bmatrix}$ $C^T = \begin{bmatrix} 1 & 4 & 16 \\ 2 & 8 & 32 \end{bmatrix}$.

(b) *Verify that $(AB)^T = B^T A^T$ where A and B are from Part (a).*

Solution: $AB = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 8 & 16 & 32 \end{bmatrix} = \begin{bmatrix} 46 & 92 & 184 \\ 17 & 34 & 68 \end{bmatrix} \quad \Rightarrow (AB)^T = \begin{bmatrix} 46 & 17 \\ 92 & 34 \\ 184 & 68 \end{bmatrix}$.

$$B^T A^T = \begin{bmatrix} 1 & 8 \\ 2 & 16 \\ 4 & 32 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 46 & 17 \\ 92 & 34 \\ 184 & 68 \end{bmatrix}.$$

Both computations gave the same result, so $(AB)^T = B^T A^T$.

Summary of properties of the determinant

(0) $\det A$ is a number determined by a square matrix A.
(1) $\det I = 1$.
(2) Adding a multiple of one row to another does not change the determinant.
(3) Multiplying a row by a number a multiplies the determinant by a.
(4) Swapping two rows reverses the sign of the determinant.
(5) $\det(AB) = \det(A)\det(B)$.
(6) A is invertible exactly when $\det A \neq 0$.

Problem 51. *Compute the determinants of the following matrices, and if the determinant is nonzero find the inverse.*

(i) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ $\quad$ *(ii)* $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ $\quad$ *(iii)* $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\quad$ *(iv)* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

Solution: (i) $\det \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = 1$; $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$.

(ii) $\det \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = 1$ (by Laplace expansion along the first column, or because the determinant of an upper-triangular matrix is the product of the diagonal entries). To find the inverse, row reduce:

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 = R_1 - bR_3]{R_2 = R_2 - cR_3} \left[\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = R_1 - aR_2} .$$

So the inverse is $\begin{bmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$.

(iii) $\det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 2$ by Laplace expansion along the top row. So we expect a 2 in the denominator of the inverse. We find the inverse by row reduction.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Swap } R_1 \text{ and } R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array}\right] \xrightarrow[R_3 = R_3/2]{R_2 = -R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

$$\xrightarrow{R_2 = R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right].$$

so the inverse is $\frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

(The original matrix was symmetric. Is it an accident that the inverse is also symmetric?)

(iv) Diagonal matrices are easy: $\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 1 \cdot 2 \cdot 3 \cdot 4 = 24.$

The inverse is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$.

Problem 52. *(Rotation matrices)*
Let $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Compute $\det R(\theta)$ *and* $R(\theta)^{-1}$.

Solution: $\det R(\theta) = (\cos(\theta))^2 + (\sin(\theta))^2 = 1$. $R(\theta)^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = R(-\theta)$.

Topic 16. Linear algebra: eigenvalues, diagonal matrices, decoupling

Problem 53. (a) *Find the eigenvalues and basic eigenvectors of* $A = \begin{bmatrix} -3 & 13 \\ -2 & -1 \end{bmatrix}$.

Solution: Characteristic equation: $\det(A - \lambda I) = \lambda^2 + 4\lambda + 29 = 0 \Rightarrow \lambda = -2 \pm 5i$.

Basic eigenvectors for λ are a basis of $\text{Null}(A - \lambda I)$.

$\lambda_1 = -2 + 5i$: $(A - \lambda_1 I) = \begin{bmatrix} -1-5i & 13 \\ -2 & 1-5i \end{bmatrix}$. Take $\mathbf{v_1} = \begin{bmatrix} 13 \\ 1+5i \end{bmatrix}$.

$\lambda_2 = -2 - 5i$: Use the complex conjgate: $\mathbf{v_2} = \overline{\mathbf{v}}_\mathbf{1} = \begin{bmatrix} 13 \\ 1-5i \end{bmatrix}$.

(b) *Find the eigenvalues and eigenvectors of* $B = \begin{bmatrix} -3 & 4 \\ 2 & -5 \end{bmatrix}$.

Solution: Characteristic equation: $|B - \lambda I| = \lambda^2 + 8\lambda + 7 = 0 \Rightarrow \lambda = -1, -7$.

Basic eigenvectors for λ are a basis of $\text{Null}(B - \lambda I)$.

$\lambda_1 = -1$: $(B - \lambda_1 I) = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}$. Take $\mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\lambda_2 = -7$: $(B - \lambda_2 I) = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$. Take $\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Problem 54. *Suppose that the matrix B has eigenvalues* 1 *and* 7, *with eigenvectors*

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

respectively.

(a) *What is the solution to* $\mathbf{x}' = Bx$ *with* $x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$?

Solution: The general solution is $\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

We use the initial condition to find c_1 and c_2:

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

In matrix form this is $\begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. So,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}.$$

So, $\mathbf{x}(t) = \frac{1}{3} e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{3} e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

(b) *Decouple the system $\mathbf{x}' = B\mathbf{x}$. That is, make a change of variables so that system is decoupled. Write the DE in the new variables.*

Solution: Decoupling is just the change of variables $\mathbf{u} = S^{-1}\mathbf{x}$. So,

$$\mathbf{u} = S^{-1}\mathbf{x} \;\Leftrightarrow\; \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \;\Leftrightarrow\; \begin{cases} u = x/6 - 5x/6 \\ v = x/6 + y/6. \end{cases}$$

In these coordinates the decoupled system is $\mathbf{u}' = \Lambda\mathbf{u} \;\Leftrightarrow\; \begin{bmatrix} u' \\ v' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$.

(c) *Give an argument based on transformations why $B = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1}$ has the eigenvalues and eigenvectors given above.*

Using the definition of eigenvalues and eigenvectors, we need to show that

$$B \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad B \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Multiplying by a standard basis vector just picks out the corresponding column of a matrix. So we have the following multiplication table:

$$\begin{aligned} S \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \Rightarrow S^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ S \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \Rightarrow S^{-1} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \Lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \Lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= 7 \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Using this table, we can now compute the product $S\Lambda S^{-1} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

$$S\Lambda S^{-1} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = S\Lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} = S \begin{bmatrix} 0 \\ 7 \end{bmatrix} = 7S \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

This shows that $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is an eigenvector of $S\Lambda S^{-1}$ with eigenvalue 7. The other eigenvalue/eigenvector pair behaves the same way.

Problem 55. *Suppose $A = \begin{bmatrix} a & b & c \\ 0 & 2 & e \\ 0 & 0 & 3 \end{bmatrix}$.*

(a) *What are the eigenvalues of A?*

Solution: For an upper triangular matrix the eigenvalues are the diagonal entries: a, 2, 3.

(b) *For what value (or values) of a, b, c, e is A singular (non-invertible)?*

Solution: $\det(A)$ = product of eigenvalues. So A is singular when $a = 0$. The parameters b, c, e can take any values.

(c) *What is the minimum rank of A (as a, b, c, e vary)? What's the maximum?*

Solution: When $a = 0$, the null space is dimension 1, so rank =2.

When $a \neq 0$, A is invertible, so has rank = 3.

(d) *Suppose $a = -5$. In the system $\mathbf{x}' = A\mathbf{x}$, is the equilibrium at the origin stable or unstable.*

Solution: The two positive eigenvalues imply the system is unstable.

Problem 56. *Suppose that* $A = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1}$.

(a) *What are the eigenvalues of A?*

Solution: The eigenvalues are the same as the diagonal matrix, i.e., 1, 2, 3.

(b) *Express A^2 and A^{-1} in terms of S.*

Solution: $A^2 = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} S^{-1}$; $\quad A^{-1} = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} S^{-1}$.

(c) *What would I need to know about S in order to write down the most rapidly growing exponential solution to $\mathbf{x}' = A\mathbf{x}$?*

Solution: You need to know the eigenvector that goes with the eigenvalue 3. That is, you need to know the third column of S.

Problem 57. We didn't cover orthogonal matrices. They won't be on the final.

(a) *An orthogonal matrix is one where the columns are orthonormal (mutually orthogonal and unit length). Equivalently, S is orthogonal if $S^{-1} = S^T$.*

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. *Find an orthogonal matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$*

Solution: The problem is asking us to diagonalize A, taking care that the matrix S is orthogonal.

A has characteristic equation: $\lambda^2 - 2\lambda - 3$. So it has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$. By inspection (or computation), we have eigenvectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

These are clearly orthogonal to each other. We normalize their lengths and use the normalized eigenvectors in the matrix S.

$$S = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}; \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \quad \Rightarrow A = S\Lambda S^{-1}.$$

Note: A is a symmetric matrix. It turns out that symmetric matrix has an orthonormal set of basic eigenvectors.

(b) *Decouple the equation* $\mathbf{x}' = A\mathbf{x}$, *with* $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Solution: The decoupling change of variable is $\mathbf{u} = S^{-1}\mathbf{x} \Leftrightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The decoupled system is $\mathbf{u}' = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{u} \Leftrightarrow \begin{cases} u_1' = -u_1 \\ u_2' = 3u_2 \end{cases}$.

Topic 17. Matrix methods for solving systems of DEs. The companion matrix

Problem 58. **(a)** *Let* $A = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$. *Solve* $\mathbf{x}' = A\mathbf{x}$.

Solution: Characteristic equation: $|A - \lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ 6 & -7-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 10 = 0$. So the eigenvalues are $\lambda = 2, -5$.

Basic Eigenvectors (basis of Null$(A - \lambda I)$):

$\lambda = 2$: $\quad A - \lambda I = \begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$\lambda = -5$: $\quad A - \lambda I = \begin{bmatrix} 9 & -3 \\ 6 & -2 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Two (modal) solutions: $\quad \mathbf{x_1}(t) = e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{x_2}(t) = e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

General solution: $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$.

Problem 59. *Solve* $x' = -3x + y, \quad y' = 2x - 2y$.

Solution: The coefficient matrix is $A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$.

Characteristic equation: $\quad \lambda^2 + 5\lambda + 4 = 0$. This has roots $\lambda = -1, -4$.

Basic eigenvectors (basis of Null$(A - \lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = -1$: $\quad A - \lambda I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$. Basic eigenvector $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\lambda = -4$: $\quad A - \lambda I = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Basic eigenvector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Modal solutions: $\mathbf{x_1}(t) = e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{x_2}(t) = e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

General solution $\mathbf{x}(\mathbf{t}) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Problem 60. ***(Complex roots)*** *Solve* $\mathbf{x}' = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} \mathbf{x}$ *for the general real-valued solution.*

Solution: Coefficient matrix: $A = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix}$.

Characteristic equation: $\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & -5 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 41 = 0$.

Eigenvalues: $\lambda = 5 \pm 4i$.

Basic eigenvectors (basis of $\text{Null}(A-\lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = 5+4i$: $A - \lambda I = \begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Basic eigenvector: $\mathbf{v} = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}$.

Complex solution:

$$\mathbf{z}(t) = e^{(5+4i)t} \begin{bmatrix} 5 \\ 2-4i \end{bmatrix} = e^{5t} \begin{bmatrix} 5\cos(4t) + i5\sin(4t) \\ 2\cos(4t) + 4\sin(4t) + i(-4\cos(4t) + 2\sin(4t)) \end{bmatrix}.$$

Both real and imaginary parts are solutions to the DE:

$$\mathbf{x_1}(t) = e^{5t} \begin{bmatrix} 5\cos(4t) \\ 2\cos(4t) + 4\sin(4t) \end{bmatrix}, \quad \mathbf{x_2}(t) = e^{5t} \begin{bmatrix} 5\sin(4t) \\ -4\cos(4t) + 2\sin(4t) \end{bmatrix}$$

General real-valued solution (by superposition): $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$.

Problem 61. *Don't dwell on the computations for this problem. Just look at the final result.*

(Repeated roots) *Solve* $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$.

Solution: The coefficient matrix is $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$.

Characteristic equation: $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = 0$

Eigenvalues: $\lambda = 2, 2$ (repeated)

Basic eigenvectors (basis of $\text{Null}(A-\lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = 2$: $A - \lambda I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Basic eigenvector $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

This gives one modal solution: $\mathbf{x_1}(t) = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Since there are not enough independent eigenvectors, the system is defective. For the second solution, we look for one of the form

$$\mathbf{x_2} = te^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\mathbf{w}.$$

($\mathbf{w}$ is called a generalized eigenvector. It satisfies $(A - 2I)\mathbf{w} = \mathbf{v}$.)

After some algebra, we find that we can take $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, $\mathbf{x_2}(t) = te^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

General solution: $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$.

Problem 62. *Solve the system $x' = x + 2y$; $y' = -2x + y$.*

Solution: The coefficient matrix is $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Characteristic equation: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0$.

Eigenvalues $1 \pm 2i$.

Basic eigenvectors (basis of $\text{Null}(A - \lambda I)$): As usual, for the 2 case, we can find eigenvectors by inspection without row reduction.

$\lambda = 1 + 2i$: $A - \lambda I = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$. Basic eigenvector $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

(We don't need an eigenvector from the complex conjugate $\lambda = 1 - 2i$.)

Complex solution: $\mathbf{z}(t) = e^{(1+2i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^t(\cos(2t) + i\sin(2t))\begin{bmatrix} 1 \\ i \end{bmatrix} = e^t\begin{bmatrix} \cos(2t) + i\sin(2t) \\ -\sin(2t) + i\cos(2t) \end{bmatrix}$.

The real and imaginary parts of $\mathbf{z}$ are both solutions:

$\mathbf{x_1}(t) = e^t\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$, $\mathbf{x_2}(t) = e^t\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$.

General solution: $\mathbf{x}(t) = c_1\mathbf{x_1} + c_2\mathbf{x_2} = c_1 e^t\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + c_2 e^t\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$.

Or $x(t) = c_1 e^t\cos(2t) + c_2 e^t\sin(2t)$; $y(t) = -c_1 e^t\sin(2t) + c_2 e^t\cos(2t)$.

Topic 20. Step and delta functions

Problem 63. *Compute the following integrals.*

(a) $\displaystyle\int_{-\infty}^{\infty} \delta(t) + 3\delta(t-2)\,dt$

Solution: Both spikes are inside the interval of integration. So the integral equals 4.

(b) $\displaystyle\int_{1}^{5} \delta(t) + 3\delta(t-2) + 4\delta(t-6)\,dt.$

Solution: Only the spike at $t = 2$ is inside the interval of integration. So the integral equals 3.

Problem 64. *Compute the following integrals.*

(a) $\displaystyle\int_{0^-}^{\infty} \cos(t)\delta(t) + \sin(t)\delta(t-\pi) + \cos(t)\delta(t-2\pi)\,dt.$

Solution: All the spikes are inside the interval of integral so the integral is

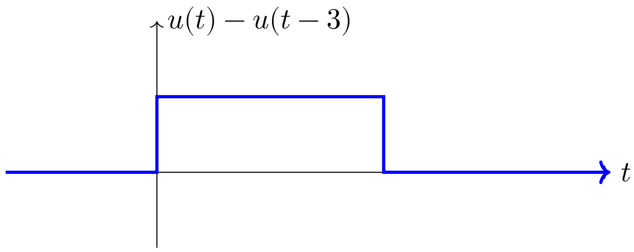
$$\cos(0) + \sin(\pi) + \cos(2\pi) = 2.$$

(b) $\int \delta(t)\,dt$. *(Indefinite integral)*

Solution: $u(t) + C$. (By definition $u' = \delta$.)

(c) $\int \delta(t) - \delta(t-3)\,dt$. *Graph the solution*

Solution: $u(t) - u(t-3) + C$.



(d) *Make up others.*

Solution: And answer them.

Problem 65. *Solve $x' + 2x = \delta(t) + \delta(t-3)$ with rest IC*

Solution: Rest IC means that for $t < 0$ we have $x(t) = 0$.

We work on the intervals between the impulses one at a time.

For $t < 0$: The DE is $x' + 2x = 0$, with $x(0^-) = 0$. The solution is $x(t) = 0$.

For $0 < t < 3$: We are given the pre-initial condition $x(0^-) = 0$.

The impulse at $t = 0$ causes x to jump by one unit. During the rest of the interval the input is 0. So, for $0 < t < 3$, we have

$$x' + 2x = 0, \quad x(0^+) = x(0^-) + 1 = 1.$$

Solving, we get $x(t) = x(0^+)e^{-2t} = e^{-2t}$.

The end condition for this interval is $x(3^-) = e^{-6}$.

For $3 < t$: From the previous interval we have the pre-initial condition $x(3^-) = e^{-6}$.

The impulse at $t = 3$ causes a unit jump in x, so $x(3^+) = x(3^-) + 1 = e^{-6} + 1$. Again, during the rest of the interval the input is 0. So we have

$$x' + 2x = 0, \quad x(3^+) = e^{-6} + 1.$$

Solving, we get $x(t) = x(3^+)e^{-2(t-3)} = (e^{-6} + 1)e^{-2(t-3)}$.

Putting the cases together, we have

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-2t} & \text{for } 0 < t < 3 \\ (e^{-6} + 1)e^{-2(t-3)} & \text{for } t > 3. \end{cases}$$

Problem 66. *(Second-order systems) Solve $4x'' + x = 5\delta(t)$ with rest IC.*

Solution: The key is understanding what jump the input $5\delta(t)$ causes at $t = 0$. In this case, since the leading coefficient is 4, $5\delta(t)$ causes a jump of 5/4 unit in x'.

Rest IC means $x(0^-) = 0$, $x'(0^-) = 0$.

For $t < 0$: The DE with initial conditions is

$$4x'' + x = 0; \qquad x(0^-) = 0,\ x'(0^-) = 0.$$

The solution on this interval is $x(t) = 0$.

For $t > 0$: The pre-initial conditions are $x(0^-) = 0, x'(0^-) = 0$. The input $5\delta(t)$ is an impulse which produces post-initial conditions

$$x(0^+) = 0, \quad x'(0^+) = x'(0^-) + 5/4 = 5/4.$$

After the impulse the input is 0. So the DE with initial conditions is

$$4x'' + x = 0; \qquad x(0^+) = 0,\ x'(0^+) = 5/4.$$

The characteristic roots are $\pm 2i$. So the general solution to the DE is

$$x(t) = c_1 \cos(t/2) + c_2 \sin(t/2).$$

We need to find c_1 and c_2 to match the post-initial conditions. The algebra yields $c_1 = 0$, $c_2 = 5/2$.

Putting the cases together, we have

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{5}{2}\sin(t/2) & \text{for } t > 0. \end{cases}$$

Problem 67. *Solve $x' + 3x = \delta(t) + e^{2t}u(t) + 2\delta(t-4)$ with rest IC.*

(The $u(t)$ is there to make sure the input is 0 for $t < 0$.)

Solution: When part of the input is a regular function, you have to organize the work carefully. Here are two ways to do it.

Method 1. Solve in cases. Each delta function adds another case.

(Case 1) $t < 0$: $x' + 3x = 0$ with rest IC $\Rightarrow x(t) = 0$.

(Case 2) $0 < t < 4$. Pre-initial conditions $x(0^-) = 0$. The delta function gives post-initial conditions: $x(0^+) = 1$.

On this interval the DE is $x' + 3x = e^{2t}$.

Using the ERF, we get the general solution $x(t) = \frac{e^{2t}}{5} + C_2 e^{-3t}$

The post-initial condition gives $x(0^+) = 1 = 1/5 + C_2$, so $\boxed{C_2 = 4/5.}$

(Case 3) $t > 4$. Using Case 2, the pre-initial conditions are $x(4^-) = \dfrac{e^8}{5} + \dfrac{4e^{-12}}{5}$. So the input $2\delta(t-4)$ gives post-initial conditions $x(4^+) = x(4^-) + 2 = \dfrac{e^8}{5} + \dfrac{4e^{-12}}{5} + 2$.

On this interval, the DE is the same as in Case 2: $x' + 3x = e^{2t}$. So we have the same general solution:

$$x(t) = \frac{e^{2t}}{5} + C_3 e^{-3t}.$$

Using the post-initial conditions, we solve for C_3:

$$\frac{e^8}{5} + C_3 e^{-12} = \frac{e^8}{5} + \frac{4e^{-12}}{5} + 2 \quad \Rightarrow \boxed{C_3 = 2e^{12} + \frac{4}{5}}.$$

Putting the cases together:

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{e^{2t}}{5} + C_2 e^{-3t} & \text{for } 0 < t < 4 \\ \frac{e^{2t}}{5} + C_3 e^{-3t} & \text{for } t > 4 \end{cases}$$

where C_2 and C_3 are boxed above.

Method 2. We find particular solutions for each of the input pieces with rest initial conditions. Then we use superposition to find the solution we want.

We do the solving for each piece very quickly. You can fill in the details.

(i) Solve $x_1' + 3x_1 = \delta(t)$ with rest IC.

For $t > 0$: The post-initial conditions are $x_1(0^+) = 1$. The DE is $x_1' + 3x_1$. We have the solution:

$$x_1(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-3t} & \text{for } t > 0. \end{cases}$$

(ii) Solve $x_2' + 3x_2 = u(t)e^{2t}$ with rest IC.

We use the exponential response formula plus the general homogeneous solution.

For $t > 0$, $x_2(t) = \dfrac{e^{2t}}{5} + Ce^{-3t}$. The initial condition is $x_2(0) = 0$. This allows us to find $C = -1/5$. So,

$$x_2(t) = \begin{cases} 0 & \text{for } t < 0 \\ , \frac{e^{2t}}{5} - \frac{e^{-3t}}{5} & \text{for } t > 0. \end{cases}$$

(iii) Solve $x_3' + 3x_3 = 2\delta(t-4)$ with rest IC.

We have pre and post-initial conditions $x_3(4^-) = 0$, $x_3(4^+) = 2$. Solving we get

$$x_3(t) = \begin{cases} 0 & \text{for } t < 4 \\ 2e^{-3(t-4)} & \text{for } t > 4. \end{cases}$$

Note: You should verify for yourself, that, since each piece satisfies the rest initial condition, so does their sum. That is, we have homogeneous IC. If the IC was inhomogeneous, we would want just one of the pieces to satisfy the inhomogeneous IC and the others to satisfy the homogeneous IC.

Problem 68. **(a)** *Solve $2x'' + 8x' + 6x = \delta(t)$ with rest IC.*

Solution: Rest IC means $x(0^-) = 0$, $x'(0^-) = 0$.

On $t < 0$: The differential equation with initial conditions is

$$2x'' + 8x' + 6x = 0; \qquad x(0^-) = 0,\ \dot{x}(0^-) = 0.$$

The solution to this is $x(t) = 0$.

On $t > 0$: The impulse at $t = 0$ causes a jump in x'. That is, we have post-initial conditions $x(0^+) = 0$, $x'(0^+) = x'(0^-) + 1/2 = 1/2$.

So the differential equation with initial conditions is

$$2x'' + 8x' + 6x = 0; \qquad x(0^+) = 0,\ \dot{x}(0^+) = 1/2.$$

The characteristic roots are -1, -3. So the general solution is

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}.$$

We find c_1 and c_2 to match the post-initial conditions: $c_1 = 0$, $c_2 = 1/2$. The complete solution is

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{e^{-t}}{4} - \frac{e^{-3t}}{4} & \text{for } 0 < t \end{cases}$$

(b) *Plug your solution into the DE and verify that it is correct*

Solution: We have to take two derivatives of x. Since $x(t)$ has no jump at $t = 0$, the generalized derivative has only a regular part.

$$x'(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{-e^{-t}}{4} + \frac{3e^{-3t}}{4} & \text{for } t > 0. \end{cases}$$

Since $x'(0^-) = 0$ and $x'(0^+) = 1/2$, x'' has a singular part:

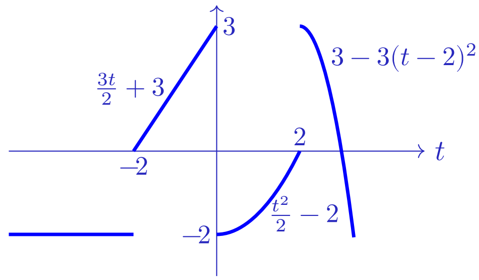
$$x''(t) = \frac{\delta(t)}{2} + \begin{cases} 0 & \text{for } t < 0 \\ \frac{e^{-t}}{4} - \frac{9e^{-3t}}{4} & \text{for } t > 0. \end{cases}$$

Thus,

$$\begin{aligned} 2x'' + 8x' + 6x &= \delta(t) + \begin{cases} 0 & \text{for } t < 0 \\ 2\left(\frac{e^{-t}}{4} - \frac{9e^{-3t}}{4}\right) + 8\left(\frac{-e^{-t}}{4} + \frac{3e^{-3t}}{4}\right) + 6\left(\frac{e^{-t}}{4} - \frac{e^{-3t}}{4}\right) & \text{for } t > 0 \end{cases} \\ &= \delta(t). \end{aligned}$$

So the solution checks out. Notice how, in the algebra, the jump of $1/2$ in x' resulted in the $\delta(t)$ term when we plugged x into the DE.

Problem 69. *The graph of the function $f(t)$ is shown below. Compute the generalized derivative $f'(t)$. Identify the regular and singular parts of the derivative.*



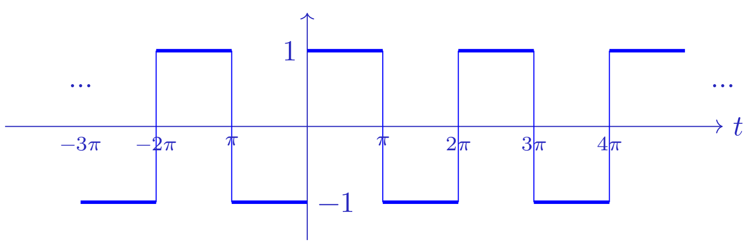
Solution: The formula for the function is

$$f(t) = \begin{cases} -2 & \text{for } t < -2 \\ \frac{3t}{2} + 3 & \text{for } -2 < t < 0 \\ \frac{t^2}{2} - 2 & \text{for } 0 < t < 2 \\ 3 - 3(t-2)^2 & \text{for } 2 < t. \end{cases}$$

We have to take the regular derivative and add delta functions at the jump discontinuities.

$$f'(t) = \underbrace{2\delta(t+2) - 5\delta(t) + 3\delta(t-3)}_{\text{singular part}} + \underbrace{\begin{cases} 0 & \text{for } t < -2 \\ \frac{3}{2} & \text{for } -2 < t < 0 \\ t & \text{for } 0 < t < 2 \\ -6(t-2) & \text{for } 2 < t. \end{cases}}_{\text{regular part}}$$

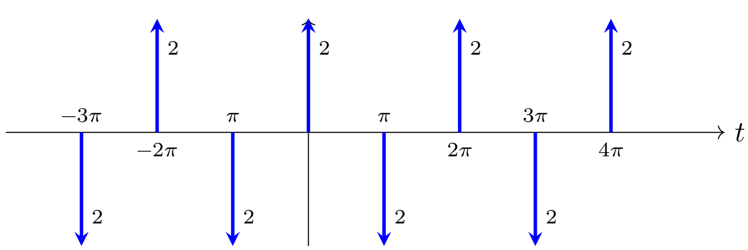
Problem 70. Derivative of a square wave
The graph below is of a function sq(t) *(called a square wave). Compute and graph its generalized derivative.*



Graph of sq(t) = *square wave*

Solution: The function alternates every π seconds between ± 1. The derivative sq$'(t)$ is clearly 0 everywhere except at the jumps. A jump of $+2$ gives a (generalized) derivative of 2δ and a jump of -2 gives a (generalized) derivative of -2δ. Thus we have

$$\text{sq}'(t) = ... + 2\delta(t+2\pi) - 2\delta(t+\pi) + 2\delta(t) - 2\delta(t-\pi) + 2\delta(t-2\pi) - 2\delta(t-3\pi) + ...$$



Graph of sq$'(t)$ = impulse train

Note that we put the weight of each delta function next to it. Conventions vary, here we used the convention that $-2\delta(t)$ is represented by a downward arrow with the weight 2 next to it. That is, the sign is represented by the direction of the arrow, so the weight is positive.

Topic 21. Fourier series
Topic 22. Fourier series continuation

Problem 71. *For each of the following:*

(i) Find the Fourier series (no integrals needed)

(ii) Identify the fundamental frequency and corresponding base frequency.

(iii) Identify the Fourier coefficients a_n and b_n

(a) $\cos(2t)$

Solution: (i) Fourier series: $\cos(2t)$.

(ii) Fundamental frequency = 2. Base period = $\dfrac{2\pi}{2} = \pi$.

(iii) $a_1 = 1$ all other coefficients are 0.

(b) $3\cos(2t - \pi/6)$

Solution: (i) Fourier series: $3\cos(\pi/6)cos(2t) + 3\sin(\pi/6)\sin(2t) = \frac{3\sqrt{3}}{2}\cos(2t) + \frac{3}{2}\sin(2t)$.

(ii) Fundamental frequency = 2. Base period = π.

(iii) $a_1 = 3\sqrt{3}/2$, $b_1 = 3/2$, all others are 0.

(c) $\cos(t) + 2\cos(5t)$

Solution: (i) Fourier series: $\cos(t) + 2\cos(5t)$.

(ii) Fundamental frequency = 1. Base period = 2π.

(iii) $a_1 = 1$, $a_5 = 2$, all others are 0.

(d) $\cos(3t) + \cos(4t)$

Solution: (i) Fourier series: $\cos(3t) + \cos(4t)$.

(ii) Fundamental frequency = 1. (Note: Every frequency must be a multiple of the fundamental frequency.) Base period = 2π. (Note: 2π is the smallest common period for all the terms.)

(iii) $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, $a_3 = 1$, $a_4 = 1$, all others are 0.

Problem 72. *Compute the Fourier series for the odd, period 2, amplitude 1 square wave. (Do this by computing integrals –not starting with the period 2π square wave.)*

Solution: $L = 1$. $f(t) = \begin{cases} -1 & \text{for } -1 \le t < 0 \\ 1 & \text{for } 0 \le t < 1 \end{cases}$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(n\frac{\pi}{L}t\right)\,dt = \left(\int_{-1}^{0} -\cos(n\pi t)\,dt + \int_0^1 \cos(n\pi t)\,dt\right) = 0.$$

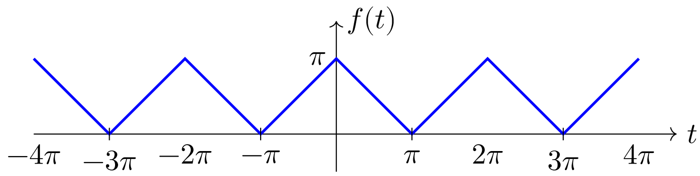
$$a_0 = \int_{-1}^{1} f(t)\,dt = 0 \text{ (by considering the area under the graph).}$$

$$b_n = \int_{-1}^{1} f(t)\sin(n\pi t)\,dt = \left(\int_{-1}^{0} -\sin(n\pi t)\,dt + \int_0^1 \sin(n\pi t)\,dt\right)$$

$$= \left[\frac{\cos(n\pi t)}{n\pi}\right]_{-1}^{0} + \left[-\frac{\cos(n\pi t)}{n\pi}\right]_0^1 = \frac{1-(-1)^n}{n\pi} - \frac{(-1)^n - 1}{n\pi} = \begin{cases} 0 & \text{if } n \text{ even} \\ 4/n\pi & \text{if } n \text{ odd} \end{cases}.$$

So, $\boxed{f(t) = \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\sin(n\pi t)}{n}.}$

Problem 73. *Compute the Fourier series for the period 2π triangle wave shown.*



Solution: $L = \pi$, $f(t) = \begin{cases} \pi + t & \text{for } -1 < t < 0 \\ \pi - t & \text{for } 0 < t < 1 \end{cases}$.

Since $f(t)$ is even, we know $b_n = 0$ and

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt = \frac{2}{\pi}\int_0^{\pi} (\pi - t)\,dt = \pi,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} (\pi - t)\cos(nt)\,dt$$

$$= \frac{2}{\pi}\left[\frac{\pi\sin(nt)}{n} - \frac{t\sin(nt)}{n} - \frac{\cos(nt)}{n^2}\right]_0^{\pi} = -\frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

$$= \begin{cases} \frac{4}{\pi n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \ne 0 \text{ even.} \end{cases}$$

So,

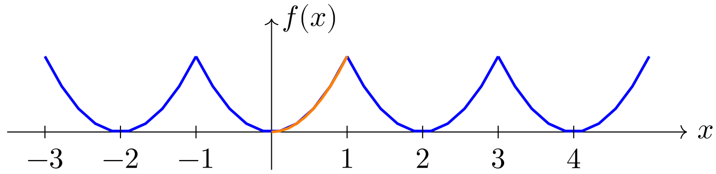
$$f(t) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\cos(nt)}{n^2}.$$

(We could also compute this using the known series for our usual triangle wave.)

Topic 23. Fourier: calculation; sine and cosine series

Problem 74. *Find the Fourier cosine series for the function $f(x) = x^2$ on $[0, 1]$. Graph the function and its even period 2 extension.*

Solution: The graph of the even, period 2 extension is shown below. $f(x)$ is shown as the orange segment above the interval $[0, 1]$.



We have $L = 1$. The cosine coefficients are computed as usual. (Or just use a table of integrals.)

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}.$$

$$\begin{aligned} a_n &= 2\int_0^1 f(x)\cos(n\pi x)\,dx = 2\int_0^1 x^2\cos(n\pi x)\,dx \\ &= 2\left[\frac{x^2\sin(n\pi x)}{n\pi} + \frac{2x\cos(n\pi x)}{(n\pi)^2} - \frac{2\sin(n\pi x)}{(n\pi)^3}\right]_0^1 \\ &= \frac{4(-1)^n}{(n\pi)^2}. \end{aligned}$$

Thus, $f(x) = \dfrac{1}{3} + 4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\pi^2 n^2}\cos(n\pi x)$.

Problem 75. *Find the Fourier series for the standard square wave shifted to the left so it's an even function, i.e., $sq(t + \pi/2)$.*

Solution: Call the standard period 2π, odd, amplitude 1 square wave $sq(t)$. We know that $sq(t) = \dfrac{4}{\pi}\displaystyle\sum_{n\ odd}\frac{\sin(nt)}{n}$

Our function is $f(t) = sq(t+\pi/2) = \dfrac{4}{\pi}\displaystyle\sum_{n\ odd}\frac{\sin(n(t+\pi/2))}{n} = \dfrac{4}{\pi}\left(\cos(t) - \frac{\cos(3t)}{3} + \frac{\cos(5t)}{5} - \ldots\right)$.

This last equation follows because

$$\sin(\theta + \pi/2) = \cos(\theta), \quad \sin(\theta + 3\pi/2) = -\cos(\theta), \quad \sin(\theta + 5\pi/2) = \cos(\theta) \ldots.$$

(You can see this either using the trig identity for $\sin(a + b)$ or by thinking about shifting a sine curve to the left by an odd multiple of $\pi/2$.)

Problem 76. *Find the Fourier sine series for $f(t) = 30$ on $[0, \pi]$.*

Solution: The sine series is the Fourier series of the odd period 2π extension of f. This is clearly our standard square wave scaled by 30. So,

$$f(t) = \frac{120}{\pi}\sum_{n\ odd}\frac{\sin(nt)}{n}.$$

Topic 24. Fourier: ODEs

Problem 77. *Solve $x' + kx = f(t)$, where $f(t)$ is the period 2π triangle wave with $f(t) = |t|$ on $[-\pi, \pi]$. (You can use the known series for $f(t)$.)*

Solution: We know the Fourier series for $f(t)$, but we'll sketch the computation.

$f(t)$ is even, so $b_n = 0$. We use the evenness to simplify the integral for the cosine coefficients

$$a_0 = \frac{2}{\pi}\int_0^\pi t\,dt = \pi, \qquad a_n = \frac{2}{\pi}\int_0^\pi t\cos(nt)\,dt = \begin{cases} -\frac{4}{\pi n^2} & \text{if } n \text{ odd} \\ 0 & \text{if } n \neq 0 \text{ even} \end{cases}$$

So the DE is: $\displaystyle x' + kx = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\,odd}\frac{\cos(nt)}{n^2}$.

Superposition: We'll solve for each piece first: $\displaystyle x_n' + kx_n = \frac{4}{n^2\pi}\cos(nt)$

We use the sinusoidal response formula (SRF). First compute $P(in)$ in polar form.

$$P(in) = k + in = \sqrt{k^2+n^2}\,e^{i\phi(n)}, \text{ where } \boxed{\phi(n) = \text{Arg}(P(in)) = \tan^{-1}(n/k) \text{ in Q1}}.$$

The SRF gives: $\displaystyle \boxed{x_{n,p}(t) = \frac{4\cos(nt - \phi(n))}{\pi n^2|P(in)|} = \frac{4\cos(nt-\phi(n))}{\pi n^2\sqrt{k^2+n^2}}}.$

Separate calculation for $n = 0$: $x_0' + kx_0 = \pi/2 \quad \Rightarrow x_{0,p}(t) = \pi/2k$.

Superposition:

$$\boxed{x_p(t) = x_{0,p} - \sum_{n\,odd} x_{n,p} = \frac{\pi}{2k} - \frac{4}{\pi}\sum_{n\,odd}\frac{\cos(nt-\phi(n))}{n^2\sqrt{k^2+n^2}}}.$$

Problem 78. *Solve $x'' + x' + 8x = g(t)$, where $g(t)$ is the period 2 triangle wave with $g(t) = |t|$ on $[-1, 1]$.*

Solution: We know that our standard, period 2π triangle wave is $\displaystyle f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\,odd}\frac{\cos(nt)}{n^2}$.

So,

$$g(t) = \frac{f(\pi t)}{\pi} = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n\,odd}\frac{\cos(n\pi t)}{n^2}.$$

(Or you can just compute the integrals for the coefficients directly.)

Use the SRF to solve for each piece: (For ease of writing, we'll leave out the coefficients here and reintroduce them in the superposition step.)

$$x_n'' + x_n' + 8x_n = \cos(n\pi t).$$

First we find $P(in)$ in polar form: $P(i\pi n) = 8 - (\pi n)^2 + i\pi n = \sqrt{(8-\pi^2n^2)^2 + \pi^2n^2}\,e^{i\phi(n)}$, where $\boxed{\phi(n) = \text{Arg}(P(in)) = \tan^{-1}(n\pi/(8-\pi^2n^2)) \text{ in Q1 or Q2.}}$

So, $x_{n,p}(t) = \dfrac{\cos(n\pi t - \phi(n))}{\sqrt{(8-\pi^2 n^2)^2 + \pi^2 n^2}}$.

Separate calculation for $n = 0$: $x_0'' + x_0' + 8x_0 = \dfrac{1}{2} \Rightarrow x_{0,p}(t) = 1/16$.

Superposition (bring back the Fourier series coefficients here):

$$x_p(t) = x_{0,p} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{x_{n,p}}{n^2} = \frac{1}{16} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{\cos(n\pi t - \phi(n))}{n^2 \sqrt{(8-\pi^2 n^2)^2 + \pi^2 n^2}}.$$

(Don't forget you need to include n in $\phi(n)$.)

Problem 79. *Solve* $x'' + 16x = \sum_{n=1}^{\infty} \dfrac{\cos(nt)}{n^2(n^2-2)^2}$. *Look out for resonance.*

Solution: Solve this in pieces: $x_n'' + 16x_n = \cos(nt)$: (We leave out the coefficient here. We'll need to include it in the superposition at the end.)

We'll need $P(in)$ in polar form.

$$P(in) = 16 - n^2 = |16-n^2| e^{i\phi(n)}, \text{ where } \phi(n) = \text{Arg}(P(in)) = \begin{cases} 0 & \text{if } n < 4 \\ \pi & \text{if } n > 4 \\ \text{undefined} & \text{if } n = 4 \end{cases}$$

Using the SRF, for $n \neq 4$, we have $x_{n,p}(t) = \dfrac{\cos(nt - \phi(n))}{|P(in)|} = \dfrac{\cos(nt-\phi(n))}{|16-n^2|}$.

For $n = 4$, we need to use the extended SRF:

$P'(r) = 2r$. So, $P'(4i) = 8i = 8e^{i\pi/2}$. Now the extended SRF gives $x_{4,p}(t) = \dfrac{t\cos(4t - \pi/2)}{8}$.

Summarizing, we have $x_{n,p}(t) = \begin{cases} \frac{\cos(nt)}{|16-n^2|} & \text{for } n < 4 \\ \frac{\cos(4t-\pi/2)}{8} & \text{for } n = 4 \\ \frac{\cos(nt-\pi)}{|16-n^2|} & \text{for } n > 4. \end{cases}$

Now, by superposition,

$$x_p(t) = \sum_{n=1}^{\infty} \frac{x_{n,p}(t)}{n^2(n^2-2)^2} = \sum_{n=1}^{3} \frac{\cos(nt)}{n^2(n^2-2)^2|16-n^2|} + \frac{t\cos(4t-\pi/2)}{8 \cdot 16 \cdot 14^2} + \sum_{n=5}^{\infty} \frac{\cos(nt-\pi)}{n^2(n^2-2)^2|16-n^2|}.$$

Finally, using $\cos(4t - \pi/2) = \sin(4t)$ and $\cos(nt - \pi) = -\cos(nt)$, we can simplify the expression for $x_p(t)$:

$$x_p(t) = \sum_{n=1}^{3} \frac{\cos(nt)}{n^2(n^2-2)^2|16-n^2|} + \frac{t\sin(4t)}{8 \cdot 16 \cdot 14^2} - \sum_{n=5}^{\infty} \frac{\cos(nt)}{n^2(n^2-2)^2|16-n^2|}.$$

Topic 25. PDEs: separation of variables

Topic 26. Continuation

Problem 80. *Let L=1. Solve the wave equation with boundary and initial conditions.*

PDE: $y_{tt} = y_{xx}$

BC: $y(0,t) = 0,\ y(1,t) = 0$

IC: $y(x,0) = 30,\ y_t(x,0) = 0.$

Solution: Step 1: Find separated solutions: $y(x,t) = X(x)T(t)$.

Plugging this into the PDE gives

$$XT'' = X''T \quad \Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \text{ constant } = -\lambda \quad \Rightarrow X'' + \lambda X = 0,\ T'' + \lambda T = 0.$$

For X, the characteristic roots are $r = \pm\sqrt{-\lambda}$. There are 3 cases:

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, $T(t) = c\cos(\sqrt{\lambda}t) + d\sin(\sqrt{\lambda}t)$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c + dt$.

Case (iii) $\lambda < 0$: Ignore, never produces nontrivial modal solutions.

Step 2: Modal solutions (separated solutions which also satisfy the BC)

For separated solutions, the BC are $X(0) = 0,\ X(1) = 0$.

We check this for our three cases.

Case (i) BC: $X(0) = a = 0$, $X(1) = a\cos(\sqrt{\lambda}) + b\sin(\sqrt{\lambda}) = 0$.

Since $a = 0$, the second condition implies either $b = 0$ or $\sin(\sqrt{\lambda}) = 0$.

If $b = 0$, then $X(x) = 0$ and all we have found is the trivial solution.

If $\sin(\sqrt{\lambda}) = 0$, then $\sqrt{\lambda} = n\pi$ for some integer n.

So, for $\lambda = \sqrt{n\pi}$, we have

$$X(x) = b\sin(n\pi x) \quad \text{and} \quad T(t) = c\cos(n\pi t) + d\sin(n\pi t).$$

Multiplying these together we get our modal solutions.

$$y_n(x,t) = \sin(n\pi x)(c_n\cos(n\pi t) + d_n\sin(n\pi t)) \quad \text{for } n = 1,\ 2, \ldots$$

(We added the index and dropped b since it was redundant.)

Case (ii) BC: $X(0) = a = 0$, $X(1) = a + b = 0$.

The only solution is $a = 0$, $b = 0$. That is, we only have the trivial solution.

Case (iii) Ignore, never produces nonzero modal solutions. (You can check is easily.)

Step 3: Using superposition, we get the general solution to the PDE + BC:

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)(c_n\cos(n\pi t) + d_n\sin(n\pi t))$$

Step 4: Use the initial conditions to determine the coefficients.

IC $y(x,0) = 30$: $y(x,0) = \sum c_n \sin(n\pi x) = 30$. This is the Fourier sine series for 30. We recognize it as the Fourier series for the odd period 2 square wave.

$$\sum c_n \sin(n\pi x) = 30 = \frac{120}{\pi}\sum_{n\ odd} \frac{\sin(n\pi x)}{n}.$$

So, $c_n = \begin{cases} \dfrac{120}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$

IC $y_t(x,t) = 0$: $y_t(x,0) = \sum n\pi d_n \sin(n\pi x) = 0$. This is a Fourier sine series for 0 on [0,1]. So, all the coefficients $n\pi d_n = 0$. This implies $d_n = 0$.

Our solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\sin(n\pi t) = \frac{120}{\pi} \sum_{n \text{ odd}} \frac{\sin(n\pi x)\sin(n\pi t)}{n}.$$

Problem 81. *Solve the heat equation with insulated ends.*

(This problem uses a cosine series, so the $\lambda = 0$ case is important.)

(PDE) $u_t = 5u_{xx}$ *for* $0 \le x \le \pi$, $t > 0$.

(BC) $u_x(0,t) = 0$, $u_x(\pi,t) = 0$

(IC) $u(x,0) = x$.

Solution: Step 1: Find the separated solutions: $u(x,t) = X(x)T(t)$

Plugging this into the DE give

$$XT' = 5X''T, \quad \frac{X''}{X} = \frac{T'}{5T} = \text{constant} = -\lambda \quad \Rightarrow X'' + \lambda X = 0,\ T' + 5\lambda T = 0.$$

For X, the characteristic roots are $\pm\sqrt{-\lambda}$. There are 3 cases.

Case (i) $\lambda > 0$: $X = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, $T = ce^{-5\lambda t}$

Case (ii) $\lambda = 0$: $X = a + bx$, $T = c$.

Case (iii) $\lambda < 0$: Ignore, this case never produces nontrivial modal solutions.

Step 2: Find modal solutions (separated solutions which also satisfy the BC).

For separated solutions, the BC are $X'(0) = 0$, $X'(\pi) = 0$.

We check the BC in each of our 3 cases.

Case (i) BC: $X'(0) = \sqrt{\lambda}b = 0$, $X'(\pi) = -a\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + b\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0$.

Since $\lambda > 0$, the first condition implies $b = 0$. Then the second condition implies either $a = 0$ or $\sin(\sqrt{\lambda}\pi) = 0$.

If $a = 0$, then $X(x) = 0$ and all we have found is the trivial solution.

If $\sin(\sqrt{\lambda}\pi) = 0$, then $\sqrt{\lambda} = n$ for some integer n.

So, for $\lambda = n$, we have $X(x) = a\cos(nx)$ and $T(t) = ce^{-5n^2 t}$.

Multiplying these together, we have the following modal solutions:

$$u_n(x,t) = a_n \cos(nx) e^{-5n^2 t} \quad \text{for } n = 1,\ 2, \ldots$$

(We added an index to name the solutions and dropped the c since it was redundant.)

Case (ii) BC: $X'(0) = b = 0$ and $X'(\pi) = b = 0$.

So, $b = 0$ and a can be any number, i.e., $X(x) = a$.

We have found one more modal solution. Knowing that the cosine series is coming, let's call it $\boxed{u_0(x,t) = a_0/2}$.

Case (iii) $\lambda < 0$: Ignore, Never produces nontrivial modal solutions.

Step 3: Using superposition, we get the general solution to PDE + BC:

$$u(x,t) = u_0(x,t) + \sum u_n(x,t) = \frac{a_0}{2} + \sum_n a_n \cos(nx) e^{-5n^2 t}$$

Step 4: Use the initial condition to determine the coefficients.

IC $u(x,0) = x$: $u(x,0) = \dfrac{a_0}{2} + \sum_n a_n \cos(nx) = x$.

This is the cosine series for x. The cosine series for x is the same as the Fourier series for our standard period 2π triangle wave. You can either remember the Fourier coefficients or compute their integrals. Either way, you get: $a_0 = \pi$, $a_n = \begin{cases} -\frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even}, n \neq 0. \end{cases}$

Thus,

$$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nx)\, e^{-5n^2 t}}{n^2}.$$

Topic 27. Qualitative behavior of linear systems, phase portraits

Problem 82. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$*, where* $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$*. What type of critical point is at the origin? Is it dynamically stable?*

Solution: First we find the eigenvalues. The characteristic equation is

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 + 4 = 0.$$

So the eigenvalues are $\pm 2i$. This means the critical point is a $\boxed{\text{center}}$.

The direction of rotation can be found by looking at the tangent vector at $(1, 0)$:

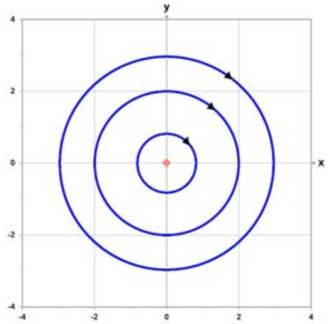
$$\mathbf{x}' = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

This tangent vector points down, which means that the ellipse is moving downwards at point $(1, 0)$ and so is moving clockwise.

Equivalently and more quickly: Because the $2, 1$ entry of A is negative, we know the trajectory turns in a $\boxed{\text{clockwise}}$ manner.

A center is on the boundary between dynamically asymptotically stable spiral sinks and dynamically unstable spiral sources. We call it an edge case. It is sometimes described as stable but not asymptotically stable.

For a center, when sketching a qualitative view of the phase portrait there is no need for eigenvectors. The trajectories are ellipses, which we have seen turn in a clockwise manner. For this system, the ellipses turn out to be perfect circles.

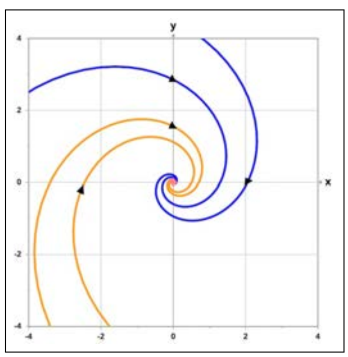


Problem 83. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$*, where* $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$*. What type of critical point is at the origin? Is it dynamically stable?*

Solution: The characteristic equation is $\lambda^2 + 2\lambda + 5 = 0$. So the eigenvalues are $-1 \pm 2i$.

Thus the critical point at the origin is a spiral sink. Since the $2, 1$ entry of A is negative. The spiral turns in a clockwise manner. Spiral sinks are dynamically stable.

For spiral sinks, a qualitative phase portrait does not require computing the eigenvectors. By hand, we would just sketch clockwise spirals, spiraling in. (Of course, the graphing program we used here is more exact.)

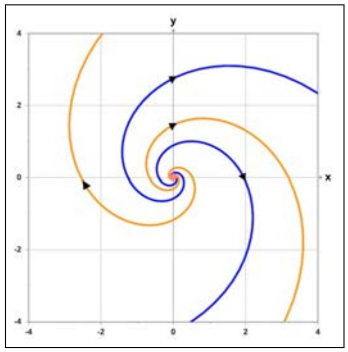


Problem 84. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$*, where* $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$*. What type of critical point is at the origin? Is it dynamically stable?*

Solution: The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$. So the eigenvalues are $1 \pm 2i$.

Thus the critical point at the origin is a spiral source. Since the $2, 1$ entry of A is negative. The spiral turns in a clockwise manner. Spiral sources are dynamically unstable.

For spiral sources, a qualitative phase portrait does not require computing the eigenvectors. By hand, we just sketch clockwise spirals, spiraling out.



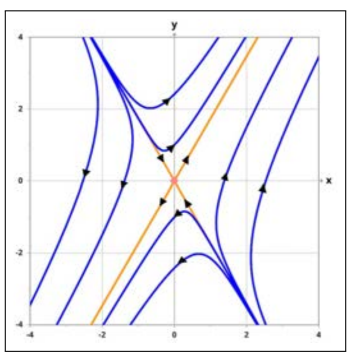
Problem 85. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$*, where* $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$*. What type of critical point is at the origin? Is it dynamically stable?*

Solution: The characteristic equation is $\lambda^2 - 2\lambda - 2 = 0$. So the eigenvalues are $1 \pm \sqrt{3}$.

Since the eigenvalues have opposite signs, the critical point at the origin is a saddle. Saddles are dynamically unstable.

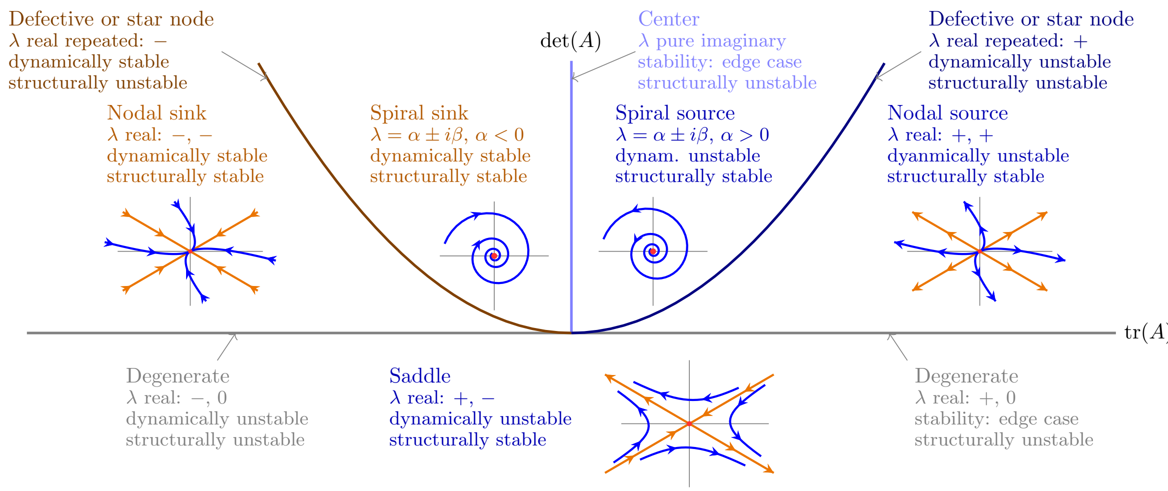
For saddles, a qualitative phase portrait requires computing the eigenvectors. We find that an eigenvector corresponding to $\lambda - 1 + \sqrt{3}$ is $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, and one corresponding to $\lambda = 1 - \sqrt{3}$ is $\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$

The modes give trajectories that are half lines. One mode has lines going out from the origin, and one has lines going in towards the origin. The mixed modal solutions are curves asymptotic to the modal trajectories at $t = \pm\infty$.



Problem 86. *Draw the trace-determinant diagram. Label all the parts with the type and dynamic stability of the critical point at the origin. Which types represent structurally stable systems?*

Solution: Here is the diagram:



The open regions in the diagram all represent structurally stable systems. That is, nodal sources, spiral sources, nodal sinks, spiral sinks and saddles are all structurally stable. The lines represent structurally unstable systems, i.e., defective and star nodes, centers, degenerate systems.

(b) *Give the equation for the parabola in the diagram. Explain where is comes from.*

Solution: The characteristic equation is $\lambda^2 - \text{tr}(A)\,\lambda + \det(A) = 0$. Therefore, the eigenvalues are

$$\lambda = \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2}.$$

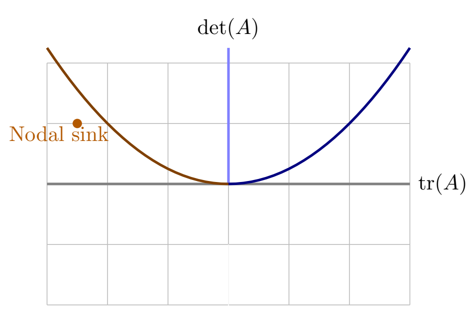
The parabola is the dividing line between real and imaginary root. That is it's where the discriminant (part under the square root) is 0. Its equation is

$$\text{tr}(A)^2 - 4\det(A) = 0 \;\;\Leftrightarrow\;\; \det(A) = \frac{\text{tr}(A)^2}{4}.$$

Problem 87. *Consider the linear system $\mathbf{x}' = A\mathbf{x}$.*

(a) *Suppose A has tr(A) $= -2.5$ and $\det(A) = 1$. Locate this system on the trace-determinant diagram. For this system, what is the type of the critical point at the origin?*

Solution: The diagram below shows the trace-determinant plane with the dividing lines included. The parabola has equation $\det(A) = \text{tr}(A)^2/4$. The point $(-2.5, 1)$ is plotted. Since it is below the parabola in the third quadrant, it represents a nodal sink.



(b) *Compute the eigenvalues of this system and verify your answer in Part (a).*

Solution: The characteristic equation is

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = \lambda^2 + 2.5\lambda + 1 = 0.$$

Therefore, the eigenvalues are $\dfrac{-2.5 \pm \sqrt{6.25 - 4}}{2} = -0.5, -2$. Since these are real and negative, the critical point at the origin is a nodal sink. This matches the answer in Part (a).

Problem 88. *For each of the following linear systems, sketch phase portraits. Give the dynamic stability of the critical point at the origin. Give the structural stability of the system.*

(a) $\mathbf{x}' = \begin{bmatrix} 5 & 1 \\ -4 & 10 \end{bmatrix}$

Solution: For all of these problems, we write the characteristic equation as

$$\lambda^2 - \mathrm{tr}(A)\,\lambda + \det(A) = 0, \qquad \text{where } A \text{ is the coefficient matrix.}$$

We only compute eigenvectors when there are real roots so we can draw the modes. For complex roots we simply determine the sense of the rotation.

We will not actually show the computations, just the results. All the pictures are at the end of the solution.

Eigenvalues: $\lambda = 6, 9$. Corresponding basic eigenvectors: $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Nodal source. The equilibrium at (0,0) is dynamically unstable. The system is structurally stable.

(b) $\mathbf{x}' = \begin{bmatrix} -7 & -3 \\ 3 & -17 \end{bmatrix}$

Eigenvalues: $\lambda = -8, -16$. Corresponding basic eigenvectors: $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Nodal sink. The equilibrium at (0,0) is dynamically asymptotically stable. The system is structurally stable.

(c) $\mathbf{x}' = \begin{bmatrix} 5 & 3 \\ 0 & -2 \end{bmatrix}$

Eigenvalues: $\lambda = 5, -2$. Corresponding basic eigenvectors: $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

Saddle. The equilibrium at (0,0) is dynamically unstable. The system is structurally stable.

(d) $\mathbf{x}' = \begin{bmatrix} 5 & 5 \\ -5 & -1 \end{bmatrix}$

Eigenvalues: $\lambda = 2 \pm 4i$.

Spiral source (clockwise). The equilibrium at (0,0) is dynamically unstable. The system is structurally stable.

(e) $\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 4 & -3 \end{bmatrix}$

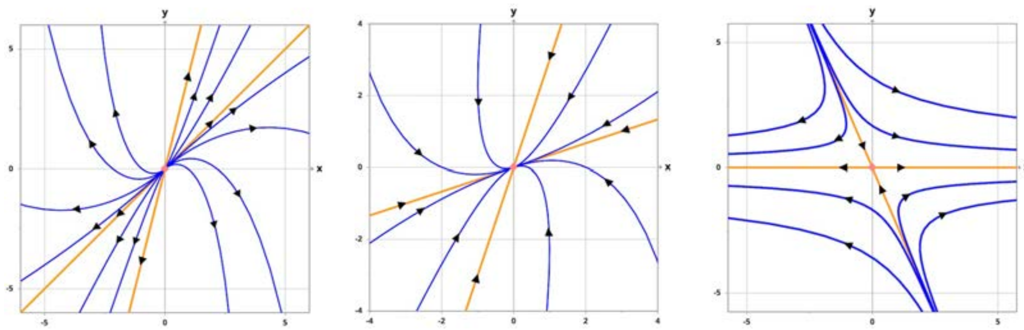
Eigenvalues: $\lambda = \pm i\sqrt{7}$.

Center (counterclockwise). The equilibrium at (0,0) is an edge case stablity-wise (or dynamically marginally stable). The system is structurally unstable.

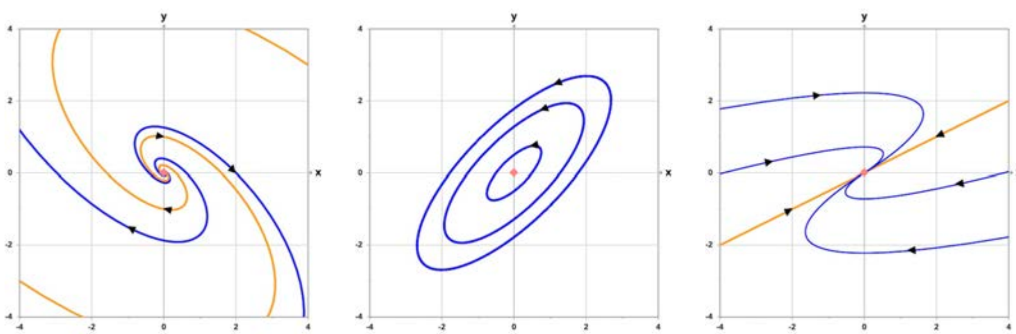
(f) $\mathbf{x}' = \begin{bmatrix} -4 & 4 \\ -1 & 0 \end{bmatrix}\mathbf{x}$

Eigenvalues: $\lambda = -2, -2$, Only one independent eigenvector $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Defective nodal sink. The equilibrium at (0,0) is dynamically asymptotically stable. The system is structurally unstable.



Parts a,b,c: nodal source, nodal sink, saddle



Parts d,e,f: spiral source, center, defective nodal sink

Topic 28. Qualitative behavior of non-linear systems
Topic 29. Structural stability

Problem 89. **(a)** *Sketch the phase portrait for* $x' = -x + xy, \quad y' = -2y + xy$.

Solution: First we find the critical points by factoring the equations:

$x' = x(-1 + y) = 0 \Rightarrow x = 0$ or $y = 1$
$y' = y(-2 + x) = 0 \Rightarrow x = 2$ or $y = 0$

So the only critical points are $(0, 0)$ and $(2, 1)$.

Jacobian: $J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -1 + y & x \\ y & -2 + x \end{bmatrix}$

At (0,0): $J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

This is the coefficient matrix A of our linearized system as (0,0). The eigenvalues are -1, -2, so this is a linearized nodal sink. Since nodal sinks are structurally stable, we also have a nonlinear nodal sink.

(As an aside, it is worth noting that the eigenvectors lie along the axes and clearly there are trajectories along each axis, i.e., if $y = 0$ the trajecory is along the x-axis.)

At (2,1): $J(2,1) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

This is the coefficient matrix A of our linearized system as (2,1). The eigenvalues are $\pm\sqrt{2}$, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

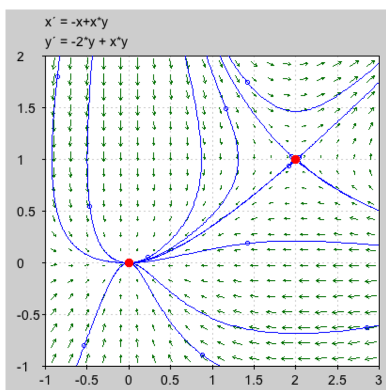
In order to sketch, we find the eigenvectors of the saddle:

The eigenvector equation is: $(A - \lambda I)\mathbf{v} = \mathbf{0}$,

$\lambda = \sqrt{2}$: $\quad A - \lambda I = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ \sqrt{2} \end{bmatrix}$.

$\lambda = -\sqrt{2}$: $\quad A - \lambda I = \begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ -\sqrt{2} \end{bmatrix}$.

Now we can sketch the linearized systems near each critical point and tie them together.



(b) *Consider x and y to be the sizes of two interacting populations. Tell a story about the populations.*

Solution: Alone each population has equation $x' = -x$ and $y' = -y$. So each would die off without the other. The interaction term xy is positive in both cases, so it seems these species cooperate to try to survive.

Unfortunately, it looks like there is a doomsday-extinction scenario. Depending on the initial conditions, Either the populations still die off to 0 (extinction) or else they explode to infinity (doomsday).

Problem 90. *Sketch the phase portrait for* $x' = x^2 - y, \quad y' = x(1-y)$.

Draw one phase portrait for each possibility for the non-structurally stable critical point.

Solution: First we find the critical points.

Factoring the second equation: $y' = x(1-y) = 0 \Rightarrow x = 0$ or $y = 1$.

Using these values in the second equation, $x' = x^2 - y = 0$ we find three critical points: $(0,0)$, $(1,1)$, $(-1,1)$.

Jacobian: $J(x,y) = \begin{bmatrix} 2x & -1 \\ 1-y & -x \end{bmatrix}$.

At $(0,0)$: $J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues are $\pm i$, so this is a linearized center. The 1 in the lower left entry of the matrix implies it turns counterclockwise.

Since centers are not structurally stable, we can't be sure the nonlinear system has a center at $(0,0)$. It could be a center, spiral source or spiral sink. We sketch all three possibilities below.

At $(1,1)$: $J(1,1) = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}$. The eigenvalues are 2, -1, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

In order to sketch, we find basic eigenvectors:

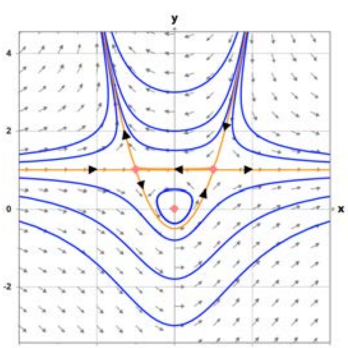
$\lambda = 2$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\quad \lambda = -1$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

At $(-1,1)$: $J(-1,1) = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}$. The eigenvalues are -2, 1, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

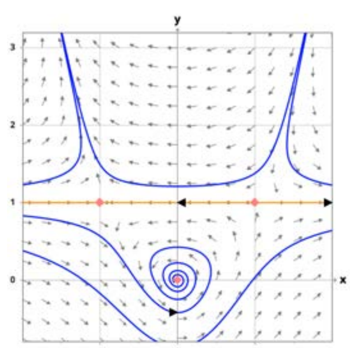
In order to sketch, we find basic eigenvectors:

$\lambda = -2$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\quad \lambda = 1$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

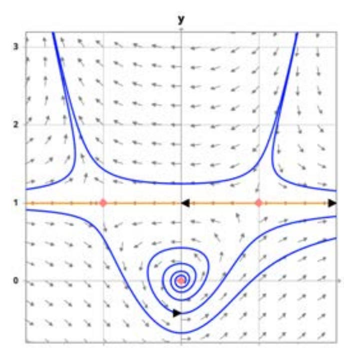
Here are sketches showing the three possible trajectories near the structurally unstable critical point.



Nonlinear center at (0,0)



Nonlinear spiral sink at (0,0)



Nonlinear spiral source at (0,0)

Problem 91. *Consider the system: $x' = x - 2y + 3, \quad y' = x - y + 2$.*

(a) *Find the one critical point and linearize at it. For the linearized system, what is the type of the critical point?*

Solution: The equations for the critical points are

$$x' = x - 2y + 3 = 0$$
$$y' = x - y + 2 = 0.$$

This is a linear system of equations. The only solution is $(x, y) = (-1, 1)$.

Jacobian: $J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

So, $J(-1, 1) = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$. Thus the linearized system at the critical point is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

This has characteristic equation $\lambda^2 + 1 = 0$. So the eigenvalues are $\pm i$. This shows the linearized system is a center.

(b) *In Part (a) you should have found that the linearized system is a center. Since this is not structurally stable, it is not necessarily true that the nonlinear system has a center at the critical point. Nonetheless, in this case, it does turn out to be a nonlinear center. Prove this.*

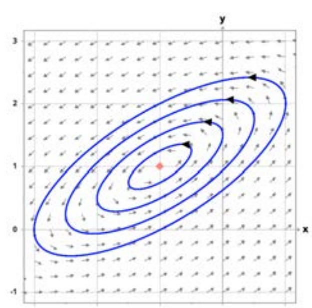
Solution: This is an inhomogeneous linear system with constant input. So one way to make a phase portrait is to solve the equation and plot trajectories.

Since the input is constant, we guess a constant solution $\mathbf{x} = \mathbf{K}$. We find $\mathbf{x_p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. (Not surprisingly this is the same as the critical point!)

The associated homogeneous system is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. For the linear, homogeneous system, the coefficient matrix has eigenvalues $\pm i$. Thus the critical point at the origin is a center.

The general solution is $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$.

Since $\mathbf{x_p}$ is a constant, the general inhomogeneous solution is just the homogeneous solution translated by $(-1, 1)$. This shows that the critical point at $(-1, 1)$ is, indeed, a center.



Problem 92. *For the following system, draw the phase portrait by linearizing at the critical points.*

$$x' = 1 - y^2, \quad y' = x + 2y.$$

Solution: First we find the critical points.

$$x' = 1 - y^2 = 0 \Rightarrow y = \pm 1$$
$$y' = x + 2y = 0 \Rightarrow x = -2y.$$

So the only critical points are $(-2, 1)$ and $(2, -1)$.

Jacobian: $J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 0 & -2y \\ 1 & 2 \end{bmatrix}$

At (-2,1): $J(-2, 1) = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$

This has eigenvalues are $1 \pm i$, so the critical point is is a linearized spiral source. The 1 in the lower left entry tells us it turns counterclockwise. Since spiral sources are structurally stable, we also have a nonlinear spiral source.

At (2,-1): $J(2, -1) = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$.

This has eigenvalues $1 \pm \sqrt{3}$, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

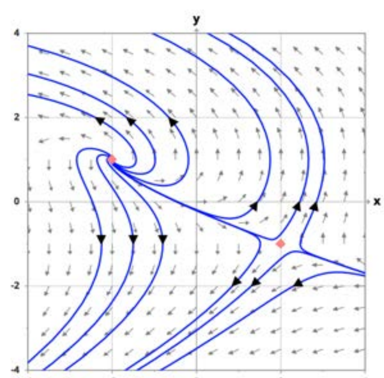
In order to sketch, we find the eigenvectors:

The eigenvector equation is: $(A - \lambda I)\mathbf{v} = \mathbf{0}$,

$\lambda = 1 + \sqrt{3}$: $\quad A - \lambda I = \begin{bmatrix} -1 - \sqrt{3} & 2 \\ 1 & 1 - \sqrt{3} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ 1 + \sqrt{3} \end{bmatrix}$.

$\lambda = 1 - \sqrt{3}$: $\quad A - \lambda I = \begin{bmatrix} -1 + \sqrt{3} & 2 \\ 1 & 1 + \sqrt{3} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ 1 - \sqrt{3} \end{bmatrix}$.

Now we can sketch the linearized systems near each critical point and tie them together.



Problem 93. *For the following system, draw the phase portrait by linearizing at the critical points.*

$$x' = x - y - x^2 + xy, \quad y' = -y - x^2.$$

Solution: First we find the critical points.

$$x' = x - y - x^2 + xy = 0$$
$$y' = -y - x^2 = 0.$$

The second equation implies $y = -x^2$. Putting this into the first equation gives

$$x + x^2 - x^2 - x^3 = x - x^3 = 0.$$

So, $x = 0,\ 1,\ -1$.

Thus the critical points are $(0,0)$, $(1,-1)$ and $(-1,-1)$.

Jacobian: $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 1-2x+y & -1+x \\ -2x & 1 \end{bmatrix}$

At (0,0): $J(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

This has eigenvalues are ± 1, so the critical point is is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

In order to sketch, we find the eigenvectors. This is straighforward, eigenvectors for $\lambda = 1, -1$ are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ respectively.

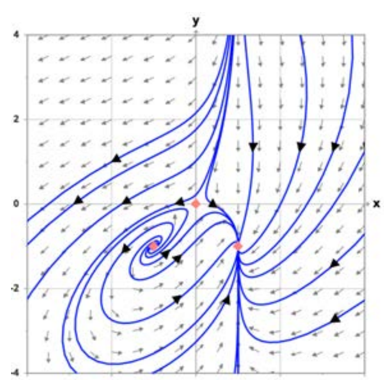
At (1,-1): $J(1,-1) = \begin{bmatrix} -2 & 0 \\ -2 & -1 \end{bmatrix}$.

This has eigenvalues -2, -1, so this is a linearized nodal sink. Since nodal sinks are structurally stable, we also have a nonlinear nodal sink.

At (-1,-1): $J(-1,-1) = \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}$.

This has eigenvalues $(1 \pm \sqrt{7}\,i)2$, so this is a linearized spiral source. The 2 in the lower left entry of the Jacobian tells us the spiral is counterclockwise. Since spiral sources are structurally stable, we also have a nonlinear spiral source.

Now we can sketch the linearized systems near each critical point and tie them together.



Topic 30. Population models

Problem 94. *Let $x(t)$ be the population of sharks off the coast of Massachusetts and $y(t)$ the population of fish. Assume that the populations satisfy the Volterra predator-prey* *equations*

$$x' = ax - pxy; \qquad y' = -by + qxy, \qquad \textit{where } a,\ b,\ p,\ q,\ \textit{are positive.}$$

Assume time is in years and a and b have units 1/years.

Suppose that, in a few years, warming waters start killing 10% of both the fish and the sharks each year. Show that the shark population will actually increase.

Solution: Original equations:

$$\begin{aligned} \text{sharks: } x' &= ax - pxy \\ \text{fish: } y' &= -by + qxy. \end{aligned}$$

The original equilibrium is (sharks, fish) = $(\frac{b}{q}, \frac{a}{p})$.

With warming:

$$\begin{aligned} x' &= (a - 0.1)x - pxy \\ y' &= -(b + 0.1)y + qxy \end{aligned}$$

The new equilibrium is (sharks, fish) = $\left(\frac{b+0.1}{q}, \frac{a-0.1}{p}\right)$. So the equilibrium level of sharks increases. (And that of fish decreases.)

Problem 95. *Consider the system of equations*

$$x'(t) = 39x - 3x^2 - 3xy; \qquad y'(t) = 28y - y^2 - 4xy.$$

The four critical points of this system are (0,0), (13,0), (0,28), (5,8).

(a) *Show that the linearized system at (0,0) has eigenvalues 39 and 28. What type of critical point is (0,0)?*

Solution: The Jacobian of the system is $J(x, y) = \begin{bmatrix} 39 - 6x - 3y & -3x \\ -4y & 28 - 2y - 4x \end{bmatrix}$.

(a) $J(0, 0) = \begin{bmatrix} 39 & 0 \\ 0 & 28 \end{bmatrix}$. This is a diagonal matrix, so the eigenvalues are the diagonal entries: $\lambda = 39,\ 28$. Positive real eigenvalues imply the linearized critical point is a nodal source. This is structurally stable, so the nonlinear critical point is also a nodal source.

(b) *Linearize the system at (13,0); find the eigenvalues; give the type of the critical point.*

Solution: $J(13, 0) = \begin{bmatrix} -39 & -39 \\ 0 & -24 \end{bmatrix}$. This is triangular, so the eigenvalues are just the diagonal entries: $\lambda = -39,\ -24$. Negative eigenvalues imply the linearized critical point is a nodal sink. This is structurally stable, so the nonlinear critical point is also a nodal sink.

(c) *Repeat Part (b) for the critical point (0,28).*

Solution: $J(0, 28) = \begin{bmatrix} -45 & 0 \\ -112 & -28 \end{bmatrix}$. This is triangular, so the eigenvalues are just the diagonal entries: $\lambda = -45,\ -28$. Negative eigenvalues imply the linearized critical point is a nodal sink. This is structurally stable, so the nonlinear critical point is also a nodal sink.

(d) *Repeat Part (b) for the critical point (5,8).*

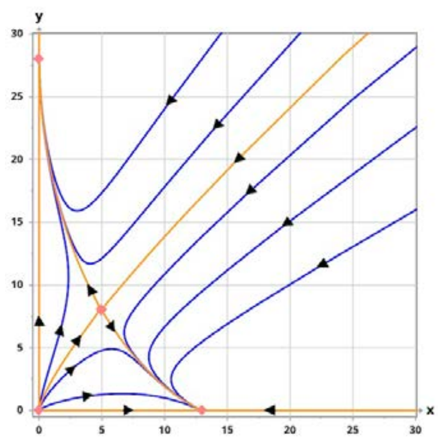
Solution: $J(5,8) = \begin{bmatrix} -15 & -15 \\ -32 & -8 \end{bmatrix}$. The characteristic equation is $\lambda^2 + 23\lambda - 360 = 0$. This has eigenvalues $\dfrac{-23 \pm \sqrt{23^2 + 4 \cdot 360}}{2}$. That is, it has one positive and one negative eigenvalue. Therefore, the linearized critical point is a saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

Note: We could also have identified this as a saddle because its determinant is negative.

(e) *Sketch a phase portrait of the system. If this models two species, what is the relationship between the species? What happens in the long-run?*

Solution: Here is the phase portrait. Note the separatix (in orange). It is made up of the trajectories that go asymptotically to the saddle point (5,8).

The relationship between the species is one of competition –you see that because both x' and y' have a $-xy$ term. In the long run one species dies out and the other stabilizes at the carrying capacity of the environment.



Problem 96. *The system for this equation is*

$$x' = 4x - x^2 - xy$$
$$y' = -y + xy$$

(a) *This models two populations with a predator-prey relationship. Which variable is the predator population?*

Solution: In the presence of y, the growth rate of x decreases. In the presence of x, the growth rate of y increases. Thus x is the prey population and y the predator population.

(b) *What would happen to the predator population in the absence of prey? What about the prey population in the absence of predators?*

Solution: Without prey, i.e., when $x = 0$, the DE for y is $y' = -y$. This is exponential decay. So eventually the predator population would go to 0.

Without predators, the equation for the prey becomes $x' = 4x - x^2$. This is the logistic equation with dynamically stable critical point $x = 4$ and dynamically unstable critical point $x = 0$. The prey population would eventially stabilize at 4.

(c) *There are three critical points. Find and classify them*

Solution: We can factor each of the equations to find the critical points:

$$x' = x(4 - x - y) = 0 \Rightarrow x = 0 \text{ or } 4 - x - y = 0$$
$$y' = y(-1 + x) \Rightarrow y = 0 \text{ or } x = 1.$$

The critical points are $(0,0)$, $(4,0)$, $(1,3)$.

The Jacobian is $J(x,y) = \begin{bmatrix} 4-2x-y & -x \\ y & -1+x \end{bmatrix}$.

Considering each of the critical points in turn:

$J(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \lambda = 4, -1.$

One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(4,0) = \begin{bmatrix} -4 & -4 \\ 0 & 3 \end{bmatrix} \Rightarrow \lambda = -4, 3.$

One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(1,3) = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + \lambda + 3 = 0 \quad \Rightarrow \lambda = -1 \pm \sqrt{11}\, i.$

Complex eigenvalues with negative real part imply this is a linearized spiral sink. This is structurally stable, so the nonlinear critical point is also a spiral sink.

(d) *Sketch a phase portrait of this system. What is the relationship between the species? What happens in the long-run?*

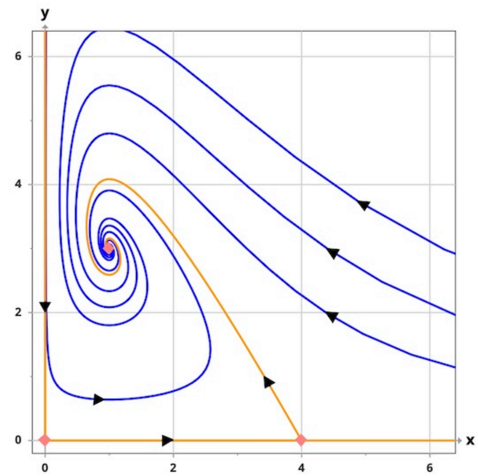
Solution: For the saddles, we need to find the eigenvectors. For the spiral, we need its direction.

$J(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$J(4,0) = \begin{bmatrix} -4 & -4 \\ 0 & 3 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \end{bmatrix}$.

The spiral at $(1,3)$ is counterclockwise because of the 3 in the lower left entry of $J(1,3)$.

Here is the phase portrait. Since we're talking about populations, the portrait only shows the first quadrant. All trajectories spiral into the critical point at (2,3). (Actually, there are a handful of trajectories along the axes that go asymptotically to the saddle points.)

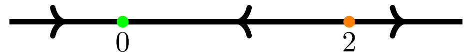


Problem 97. *The equations for this system are*

$$x' = x^2 - 2x - xy$$
$$y' = y^2 - 4y + xy$$

(a) *If this models two populations, what would happen to each of the populations in the absence of the other?*

Solution: If $y(t) = 0$, then $x' = x^2 - 2x$. This has critical points $x = 0, 2$ and phase line



So, without any predator ($y(t) = 0$), the prey population x will either crash to 0 or boom to infinity –at least according to this model.

The answer is the same for $y(t)$ if $x(t) = 0$.

(b) *There are four critical points. Find and classify them*

Solution: Again, we can factor to find the critical points.

$$x' = x(x - 2 - y) = 0 \Rightarrow x = 0 \text{ or } x - 2 - y = 0$$
$$y' = y(y - 4 + x) = 0 \Rightarrow y = 0 \text{ or } y - 4 - x = 0.$$

First let $x = 0$, then $y = 0$ or $y = 4$: two critical points (0,0), (0,4).

Next let $y = 0$, then $x = 0$ or $x = 2$: one more critical point (2,0).

Finally, solve $x - 2 - y = 0$, $y - 4 - x = 0$: one more critical (3,1).

The Jacobian is $J(x, y) = \begin{bmatrix} 2x - 2 - y & -x \\ y & 2y - 4 + x \end{bmatrix}$. Looking at each critical point in turn we get

$J(0, 0) = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \Rightarrow \lambda = -2, -4$. Negative eigenvalues imply this is a linearized nodal sink. This is structurally stable, so the nonlinear critical point is also a nodal sink.

$J(0, 4) = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix} \Rightarrow \lambda = -6, 4$. One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(2,0) = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix} \quad \Rightarrow \lambda = 2, -2$. One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(3,1) = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$.

Characteristic equation: $\lambda^2 - 4\lambda + 6 = 0 \;\Rightarrow \lambda = 2 \pm \sqrt{2}\, i$ Complex eigenvalues with positive real part imply this is a linearized spiral source. This is structurally stable, so the nonlinear critical point is also a spiral source.

(c) *Sketch a phase portrait of the system. What is the relationship between the species? What happens in the long-run?*

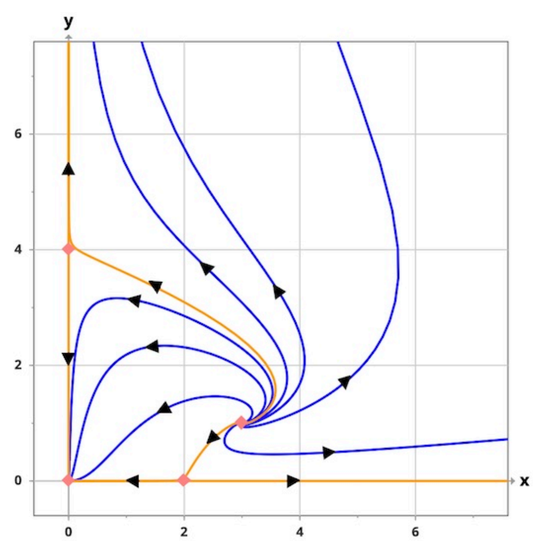
Solution: For the saddles, we need to find the eigenvectors. For the spiral, we need its direction.

$J(0,4) = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$J(2,0) = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The spiral at $(1,3)$ is counterclockwise because of the 1 in the lower left entry of $J(1,3)$.

Here is the phase portrait. The trajectories either go asymptotically to $(0,0)$ or to ∞. This looks like a predator-prey relationship. What seems more important, is that each population by itself is modeled by a doomsday-extinction equation. That is, either the population goes to ∞ or to 0. It's hard to tell exactly, but it seems that when the predator (y) goes to infinity, the prey (x) goes extinct.



Topic 31. Physical models: the pendulum

We didn't cover physical models in class. This is still good practice for creating and interpreting phase portraits.

Problem 98. **Nonlinear Spring**
The following DE models a nonlinear spring:

$$m\ddot{x} = -kx + cx^3 \quad \begin{cases} \textbf{hard if } c < 0 & \text{(cubic term adds to linear force)} \\ \textbf{soft if } c > 0 & \text{(cubic term opposes linear force).} \end{cases}$$

(a) *Convert this to a companion system of first-order equations.*

Solution: The companion system is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -kx/m + cx^3/m \end{aligned}$$

(b) *Sketch a phase portrait of the system for both the hard and soft springs. You can use the fact that the linearized centers are also nonlinear centers. (This follows from energy considerations.)*

Solution: Case 1. Hard spring ($c < 0$): One critical point at $(0, 0)$

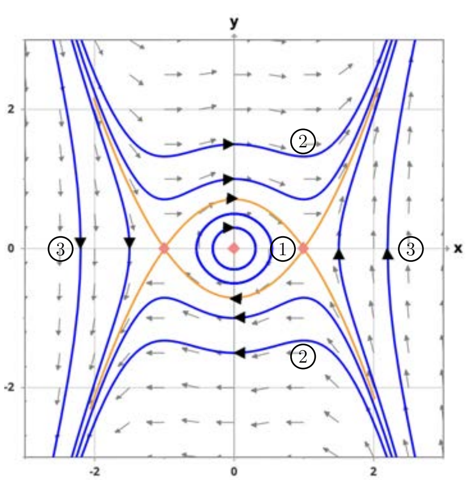
The Jacobian $J(x, y) = \begin{bmatrix} 0 & 1 \\ -k/m + 3cx^2/m & 0 \end{bmatrix}$

$J(0, 0) = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \Rightarrow \lambda = i\sqrt{k/m}$. So we have a linearized center. The problem statement tells us that this is also a nonlinear center.

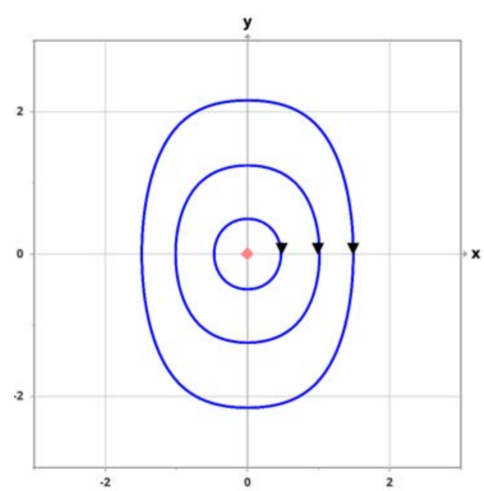
Case 2. Soft spring ($c > 0$): We have the following critical points: $(0, 0)$, $(\pm\sqrt{k/c}, 0)$.

$(0, 0)$: $J(0, 0)$ is the same as for the hard spring. This is a linearized center. The problem statement says it is also a nonlinear center.

$(\pm\sqrt{k/c}, 0)$: $J(\pm\sqrt{k/c}, 0) = \begin{bmatrix} 0 & 1 \\ 2k/m & 0 \end{bmatrix}$ (same for both). Thus we have linearized saddles and, by structural stability, nonlinear saddles. (You should find the eigenvectors to aid in sketching the phase portrait.)



Soft spring: $c > 0$



Hard spring: $c < 0$

(c) *(Challenge! For anyone who is interested. This is not part of the ES.1803 syllabus.) Find equations for the trajectories of the system.*

Solution: We use a standard trick to get trajectories:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-kx + cx^3}{my}.$$

This is separable: $my\,dy = (-kx + cx^3)\,dy$. Integrating we get

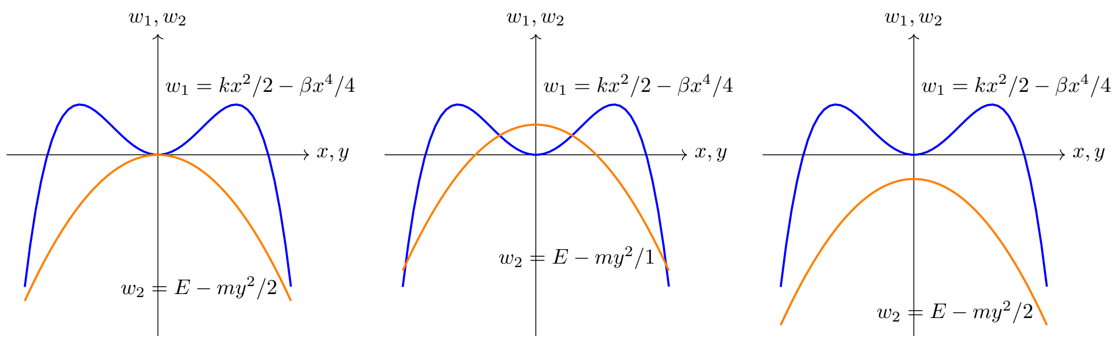
$$\underbrace{\frac{my^2}{2}}_{\text{kinetic energy}} + \underbrace{\frac{kx^2}{2} - \frac{cx^4}{4}}_{\text{potential energy}} = \underbrace{E}_{\text{total energy = constant}}.$$

If $c < 0$ (hard spring), then both energy terms on the right are positive, so x and y must be bounded. Then, for fixed x, there are at most two points on the trajectory. Thus we must have closed trajectories.

If $c > 0$ (soft spring), then, we can define w_1 and w_2 by

$$w_1(x) = \frac{kx^2}{2} - \frac{cx^4}{4}, \quad w_2(y) = E - \frac{my^2}{2}$$

Using $k > 0$, $m > 0$, we have the graphs of w_1, w_2 given below. Using the same graphical ideas as in the proof in the Topic 30 notes that the Volterra predator-prey equation has closed trajectories, this shows the phase plane for the soft spring is as shown above.



Plots of $w_1 = \frac{kx^2}{2} - \frac{cx^4}{4}$, $w_2 = E - y^2$

Different energy levels correspond to different types of trajectories. At the unstable equilibrium we compute $E = \frac{k^2}{4c}$. We have the following correspondence between energy level and trajector (using the labels on the soft-spring phase portrait above):

$E = 0$: Stable equilibrium.

$0 < E < \dfrac{k^2}{4c}$: Trajectories 1.

$E = \dfrac{k^2}{4c}$: Unstable equilibrium, or a trajectory going asymptotically to or from the unstable equilibrium.

$\dfrac{k^2}{4c} < E$: Trajectories 2.

$E < \dfrac{k^2}{4c}$ (including $E < 0$): Trajectories 3

Problem 99. **Nonlinear Spring**

The following DE models a nonlinear spring:

$$m\ddot{x} = -kx + cx^3 \quad \begin{cases} \textbf{\textit{hard if }} c < 0 & \textit{(cubic term adds to linear force)} \\ \textbf{\textit{soft if }} c > 0 & \textit{(cubic term opposes linear force).} \end{cases}$$

(a) *Convert this to a companion system of first-order equations.*

Solution: The companion system is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -kx/m + cx^3/m \end{aligned}$$

(b) *Sketch a phase portrait of the system for both the hard and soft springs. You can use the fact that the linearized centers are also nonlinear centers. (This follows from energy considerations.)*

Solution: Case 1. Hard spring ($c < 0$): One critical point at $(0,0)$

The Jacobian $J(x,y) = \begin{bmatrix} 0 & 1 \\ -k/m + 3cx^2/m & 0 \end{bmatrix}$

$J(0,0) = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \Rightarrow \lambda = i\sqrt{k/m}$. So we have a linearized center. The problem statement tells us that this is also a nonlinear center.

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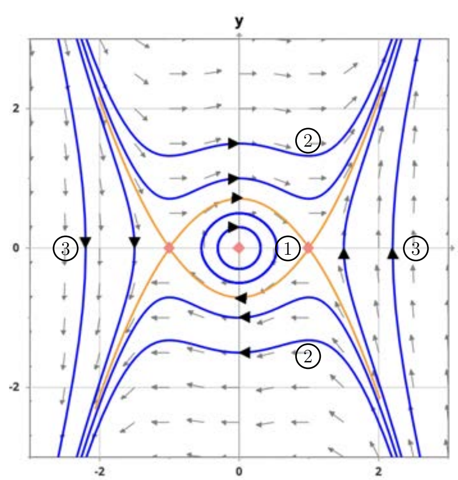
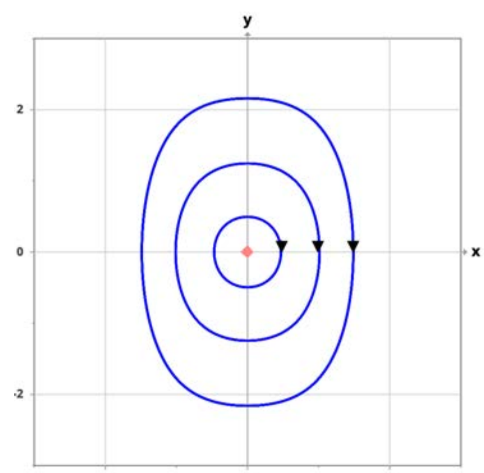


Soft spring: $c > 0$



Hard spring: $c < 0$

(c) *(Challenge! For anyone who is interested. This is not part of the ES.1803 syllabus.) Find equations for the trajectories of the system.*

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$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-kx + cx^3}{my}.$$

This is separable: $\quad my\,dy = (-kx + cx^3)\,dy$. Integrating we get

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If $c > 0$ (soft spring), then, we can define w_1 and w_2 by

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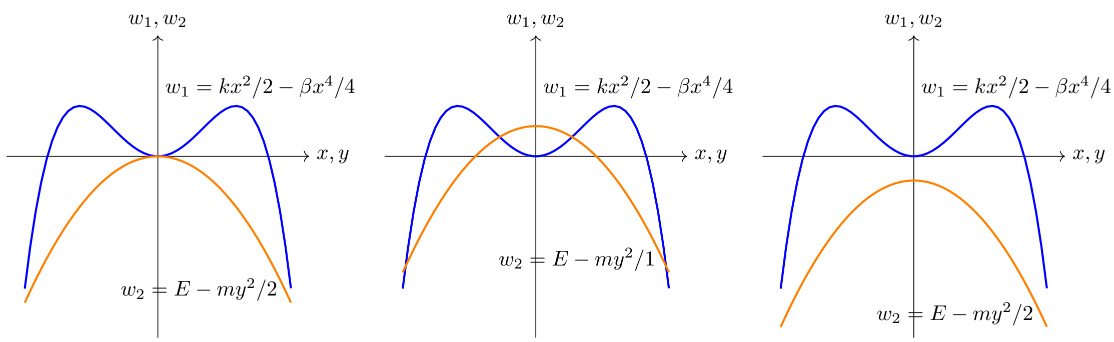

Plots of $w_1 = \frac{kx^2}{2} - \frac{cx^4}{4}, \quad w_2 = E - y^2$

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$\dfrac{k^2}{4c} < E$: Trajectories 2.

$E < \dfrac{k^2}{4c}$ (including $E < 0$): Trajectories 3