Review of entire semester, Spring 2024

This is a large set of problems covering all the topics. Most are taken from the problem section worksheets.

Topic 1. Modeling; separable DEs

Problem 1. (Here's the second geometry example in the notes for Topic 1.)

$y = y(x)$ is a curve in the first quadrant. The part of the tangent line in the first quadrant is bisected by the point of tangency. Find and solve the DE for this curve.

Problem 2. Consider the family of all lines whose y-intercept is twice the slope.

(a) Find a DE which has this family as its solutions.

(b) Find the orthogonal trajectories to the curves in Part (a). That is, find a family of functions whose graphs intersect all the lines in Part (a) orthogonally.

(c) Sketch both families.

Problem 3. You deposit money in a bank at the rate of \$1000/year. The money earns (continuous) 8% interest. Construct a DE to model the amount of money in the bank as a function of time; then solve the DE. Assume that at time 0 there is no money in the bank.

Topic 2. Linear DEs

Problem 4. (Linear homogeneous)
(a) Solve $y' + ky = 0$.

(b) Solve $y' + ty = 0$.

Problem 5. Solve $y' + ty = t^3$. (Hint: use Part (b) of the previous problem.

Problem 6. (a) Solve $y' + 2y = 2$.

(b) Solve $y' + 2y = 2t$.

(c) Solve $y' + 2y = 5 + 2t$.

Problem 7. (IVP using definite integrals)
Solve $xy' - e^x y = 0, \quad y(1) = 2$ using definite integrals.

Problem 8. Solve $y' + 2y = 2$; $y(1) = 1$.

Problem 9. Show that $y' + y^2 = q$ does not satisfy the superposition principle.

Topic 3. Input response models

Problem 10. Solve the DE $x' + 2x = f(t)$, $x(0) = 0$, where $f(t) = \begin{cases} 6 & \text{for } 0 \leq t < 1 \\ 0 & \text{for } 1 \leq t < 2 \\ 6 & \text{for } 2 \leq t. \end{cases}$

Topic 4. Complex arithmetic and exponentials

Problem 11. Polar coordinates: Write $z = -2 + 3i$ in polar form.

Problem 12. Write $3e^{i\pi/6}$ in rectangular coordinates.

Problem 13. (Trig triangle)
Draw and label the triangle relating rectangular with polar coordinates.

Problem 14. Compute $\dfrac{1}{-2+3i}$ in polar form. Convert the denominator to polar form first. Be sure to describe the polar angle precisely.

Problem 15. Find a formula for $\cos(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

Problem 16. (Roots)
Find all fifth roots of -2. Give them in polar form. Draw a figure showing the roots in the complex plane.

Problem 17. Compute $I = \displaystyle\int e^{2x} \cos(3x)\, dx$ using complex techniques.

Problem 18. (a) Show $\cos(t) = (e^{it} + e^{-it})/2$ and $\sin(t) = (e^{it} - e^{-it})/2i$.
(b) Find all the real-valued functions of the form $\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it}$.

Problem 19. Find all the real-valued functions of the form $x = \tilde{c}e^{(2+3i)t}$.

Problem 20. Find the 3 cube roots of 1 by locating them on the unit circle and using basic trigonometry.

Problem 21. Express in the form $a + bi$ the 6 sixth roots of 1.

Problem 22. Use Euler's formula to derive the trig addition formulas for sin and cos.

Topic 5. Constant coefficient linear homogeneous DEs; Damping

Problem 23. (a) Solve $x'' - 8x' + 7x = 0$ using the characteristic equation method.
(b) Solve $x'' + 2x' + 5x = 0$ using the characteristic equation method.
(c) Assume the polynomial $r^5 + a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0$ has roots

$$0.5, \quad 1, \quad 1, \quad 2 \pm 3i.$$

Give the general real-valued solution to the homogeneous constant coefficient DE

$$x^{(5)} + a_4 x^{(4)} + a_3 x^{(3)} + a_2 x'' + a_1 x' + a_0 x = 0.$$

Problem 24. (Unforced second-order physical systems)
The DE $x'' + bx' + 4x = 0$ models a damped harmonic oscillator. For each of the values $b = 0, 1, 4, 5$ say whether the system is undamped, underdamped, critically damped or overdamped.

Sketch a graph of the response of each system with initial condition $x(0) = 1$ and $x'(0) = 0$. (It is not necessary to find exact solutions to do the sketch.)

Say whether each system is oscillatory or non-oscillatory.

Problem 25. State and verify the superposition principle for $mx'' + bx' + kx = 0$, (m, b, k constants).

Problem 26. A constant coefficient, linear, homogeneous DE has characteristic roots

$$-1 \pm 2i,\ -2,\ -2, -3 \pm 4i.$$

(a) What is the order of the DE? (Notice the $\pm$ in the list of roots.)

(b) What is the general, real-valued solution.

(c) Draw the pole diagram for this system. Explain why it shows that all solutions decay exponentially to 0. What is the exponential decay rate of the general solution?

Topic 6. Exponential Response Formula

Problem 27. Let $P(D) = D^2 + 8D + 7$. Find the general real-valued solution to the following.

For oscillatory answers your particular solutions should be in amplitude-phase form.

(a) $P(D)x = e^{2t}$.

(b) $P(D)x = \cos(3t)$.

(c) $P(D)x = e^{2t}\cos(3t)$.

(d) $P(D)x = e^{-t}$.

Topic 7. Undetermined coefficients; Theory

Problem 28. Find the general solution to $x' + 3x = t^2 + 3$

Problem 29. Find one solution to $x''' + 3x'' + 2x' + 5x = 4$.

Problem 30. Find the general solution to $x'' + 3x' = t + 1$.

Topic 8. Stability

Stability is about the system not the input.

Problem 31. Is the system $x'' + x' + 4x = 0$ stable?

Problem 32. Is a 4th order system with roots $\pm 1, -2 \pm 3i$ stable. Which solutions to the homogeneous DE go to 0 as $t \to \infty$?

Problem 33. For what k is the system $x' + kx = 0$ stable?

Topic 9. Amplitude response, resonance and practical resonance

Problem 34. Consider the system $x'' + 8x = F_0 \cos(\omega t)$.

(a) Why is this called a driven undamped system?

(b) Solve this using the sinusoidal response formula (SRF). Then do it again using complex replacement and the exponential response formula (ERF).

(c) Consider the right hand side of the DE to be the input. Graph the amplitude response function.

(d) What is the resonant frequency of the system?

(e) Why is this called the natural frequency?

Problem 35. Consider the forced damped system: $x'' + 2x' + 9x = \cos(\omega t)$.

(a) What is the natural frequency of the system?

(b) Find the response of the system in amplitude-phase form.

(c) Consider the right hand side of the DE to be the input. What is the amplitude response of the system? Draw its graph –be sure to label your axes correctly

(d) What is the practical resonant frequency?

(e) When $\omega = \sqrt{7}$ by how many radians does the output peak lag behind the input peak?

(f) For the forced undamped system $x'' + 9x = \cos(\omega t)$ give a detailed description of the phase lag for different input frequencies?

Problem 36. Consider the driven first-order system: $x' + kx = kF_0 \cos(\omega t)$. We'll take the input to be $F_0 \cos(\omega t)$. Solve the DE. Find the amplitude response. Show there is never practical resonance.

Topic 10. Direction fields, integral curves
Topic 11. Numerical methods: Euler's method

Problem 37. Consider $y' = x^2 - y^2$

(a) Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as + and negative slope as −. Use this to give a very rough sketch of some solution curves.

Note: the nullcline consists of two lines.

(b) Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.

(c) Add some integral curves to the plot in Part (b). Include the one with $y(2) = 0$.

(d) Use squeezing to estimate $y(100)$ for the solution with IC $y(2) = 0$.

(e) Use Euler's method with $h = 0.5$ to estimate $y(3)$ for the solution with $y(2) = 0$.

(f) Is the estimate in Part (e) too high or too low?

Topic 12. Autonomous first-order DEs

Problem 38. Let $x' = x(x-a)(x-3)$

(a) Let $a = 1$ draw phase line. Identify type of each critical point, sketch solution graphs.

(b) Considering a to be a parameter: draw the bifurcation diagram: identify the stable and unstable branches.

(c) If this models a population, for what a is the population sustainable?

Topic 13. Linear algebra: linearity, vector spaces, connection to DEs
Topic 14. Linear algebra: row reduction, column space, pivots

Problem 39. Solve this system of linear equations. How many methods can you think of to solve this system?

$$\begin{aligned} x + y &= 5 \\ 3x + 2y &= 7 \end{aligned}$$

Problem 40. Let $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Suppose R is the row reduced echelon form for A.

(a) What is the rank of A?

(b) Find a basis for the null space of A.

(c) Suppose the column space of A has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Find a possible matrix for A. That is, give a matrix A with RREF R and the given column space.

(d) Find a matrix with the same row reduced echelon form, but such that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are in its column space.

Problem 41. Consider the following system of equations:

$$\begin{aligned} x + \ \ y + \ \ z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned}$$

(a) Write this system of equations as a matrix equation.

(b) Use row reduction to get to row echelon form. What is the solution set?

Problem 42. Solve the following equation using row reduction:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) At the end of the row-reduction process, was the last column pivotal or free? Is this related to the absence of solutions?

(b) Find a new vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ such that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has a solution.

Problem 43. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & 6 & 2 & 4 \\ 0 & 0 & 10 & 3 & 6 \end{bmatrix}$. Put A in row reduced echelon form. Find the column space, null space, rank, a basis for the column space, a basis for the null space, the dimension of each of the spaces.

Problem 44. **(a)** Suppose we have a matrix equation

$$\begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & x \end{bmatrix}$$

Can you specify x? For any value of x you think is allowable, find such an equation. Can any of the $\bullet$'s be 0?

(b) Suppose we have a matrix equation

$$\begin{bmatrix} \bullet & 3 \\ \bullet & 4 \\ \bullet & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Can you specify the $\bullet$'s?

(c) Suppose we have a matrix equation

$$\begin{bmatrix} x & 3 \\ y & 4 \\ z & 5 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$?

Problem 45. Suppose we have a matrix equation

$$\begin{bmatrix} 1 & x & 2 \\ 3 & y & 4 \\ 5 & z & 6 \end{bmatrix} \mathbf{c} = \mathbf{0}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$?

Problem 46. For what values of y is it the case that the columns of $\begin{bmatrix} 1 & 1 & 2 \\ 3 & y & 4 \\ 5 & 1 & 6 \end{bmatrix}$ form a linearly independent set?

Problem 47. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$:

(a) Find the row reduced echelon form of A; call it R.

(b) The last column of R should be a linear combination of the first columns in an obvious way. This is a linear relation among the columns of R. Find a vector $\mathbf{x}$, such that $R\mathbf{x} = \mathbf{0}$, which expresses this linear relationship.

(c) Verify that the same relationship holds among the columns of A.

(d) Explain why the linear relations among the columns af R are the same as the linear relations among the columns of A. In fact, explain why, if A and B are related by row transformations, the linear relations among the columns of A are the same as the linear relations among the columns of B.

Problem 48. This continues the previous problem. Now, suppose we want to solve $A\mathbf{x} = \mathbf{b}$, where, again, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

(a) When is this possible? Answer this in the form: "**b** must be a linear combination of the two vectors …"

(b) $A\mathbf{x} = \mathbf{b}$ is certainly solvable for $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. (What is the obvious particular solution?) Describe the general solution to this equation, as $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$.

Problem 49. Suppose that the row reduced echelon form of the 4×6 matrix B is

$$R = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a linearly independent set of vectors of which every vector in the null space of B is a linear combination.

(b) Write the columns of B as $\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_6}$. What is $\mathbf{b_1}$? What can we say about $\mathbf{b_2}$? Which of these vectors are linearly independent of the preceding ones? Express the ones which are not independent as explicit linear combinations of the previous ones. Describe a linearly independent set of vectors of which every vector in the column space of B is a linear combination.

Topic 15. Linear algebra: transpose, inverse, determinant

Problem 50. Compute the transpose of the following matrices.

(a) $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 2 & 4 \\ 8 & 16 & 32 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 2 \\ 4 & 8 \\ 16 & 32 \end{bmatrix}$

(b) Verify that $(AB)^T = B^T A^T$ where A and B are from Part (a).

Summary of properties of the determinant

(0) $\det A$ is a number determined by a square matrix A.
(1) $\det I = 1$.
(2) Adding a multiple of one row to another does not change the determinant.
(3) Multiplying a row by a number a multiplies the determinant by a.
(4) Swapping two rows reverses the sign of the determinant.
(5) $\det(AB) = \det(A)\det(B)$.
(6) A is invertible exactly when $\det A \neq 0$.

Problem 51. Compute the determinants of the following matrices, and if the determinant is nonzero find the inverse.

(i) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

Problem 52. (Rotation matrices)
Let $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
Compute $\det R(\theta)$ and $R(\theta)^{-1}$.

Topic 16. Linear algebra: eigenvalues, diagonal matrices, decoupling

Problem 53. (a) Find the eigenvalues and basic eigenvectors of $A = \begin{bmatrix} -3 & 13 \\ -2 & -1 \end{bmatrix}$.

(b) Find the eigenvalues and eigenvectors of $B = \begin{bmatrix} -3 & 4 \\ 2 & -5 \end{bmatrix}$.

Problem 54. Suppose that the matrix B has eigenvalues 1 and 7, with eigenvectors

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

respectively.

(a) What is the solution to $\mathbf{x}' = Bx$ with $x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$?

(b) Decouple the system $\mathbf{x}' = B\mathbf{x}$. That is, make a change of variables so that system is decoupled. Write the DE in the new variables.

(c) Give an argument based on transformations why $B = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1}$ has the eigenvalues and eigenvectors given above.

Problem 55. Suppose $A = \begin{bmatrix} a & b & c \\ 0 & 2 & e \\ 0 & 0 & 3 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) For what value (or values) of a, b, c, e is A singular (non-invertible)?

(c) What is the minimum rank of A (as a, b, c, e vary)? What's the maximum?

(d) Suppose $a = -5$. In the system $\mathbf{x}' = A\mathbf{x}$, is the equilibrium at the origin stable or unstable.

Problem 56. Suppose that $A = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1}$.

(a) What are the eigenvalues of A?

(b) Express A^2 and A^{-1} in terms of S.

(c) What would I need to know about S in order to write down the most rapidly growing exponential solution to $\mathbf{x}' = A\mathbf{x}$?

Problem 57. We didn't cover orthogonal matrices. They won't be on the final.

(a) An orthogonal matrix is one where the columns are orthonormal (mutually orthogonal and unit length). Equivalently, S is orthogonal if $S^{-1} = S^T$.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find an orthogonal matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$

(b) Decouple the equation $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Topic 17. Matrix methods for solving systems of DEs. The companion matrix

Problem 58. **(a)** Let $A = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$. Solve $\mathbf{x}' = A\mathbf{x}$.

Problem 59. Solve $x' = -3x + y, \quad y' = 2x - 2y$.

Problem 60. (**Complex roots**) Solve $\mathbf{x}' = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} \mathbf{x}$ for the general real-valued solution.

Problem 61. *Don't dwell on the computations for this problem. Just look at the final result.*

(**Repeated roots**) Solve $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$.

Problem 62. Solve the system $x' = x + 2y; \quad y' = -2x + y$.

Topic 20. Step and delta functions

Problem 63. Compute the following integrals.

(a) $\displaystyle\int_{-\infty}^{\infty} \delta(t) + 3\delta(t-2)\,dt$

(b) $\displaystyle\int_1^5 \delta(t) + 3\delta(t-2) + 4\delta(t-6)\,dt.$

Problem 64. Compute the following integrals.

(a) $\displaystyle\int_{0^-}^{\infty} \cos(t)\delta(t) + \sin(t)\delta(t-\pi) + \cos(t)\delta(t-2\pi)\,dt.$

(b) $\displaystyle\int \delta(t)\,dt.$ (Indefinite integral)

(c) $\displaystyle\int \delta(t) - \delta(t-3)\,dt.$ Graph the solution

(d) Make up others.

Problem 65. Solve $x' + 2x = \delta(t) + \delta(t-3)$ with rest IC

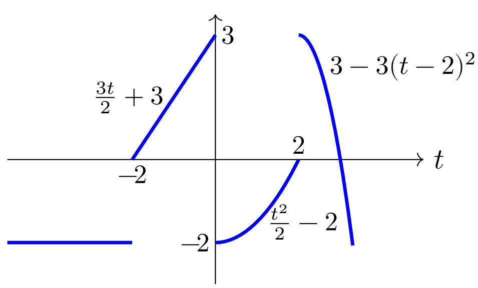
Problem 66. (Second-order systems) Solve $4x'' + x = 5\delta(t)$ with rest IC.

Problem 67. Solve $x' + 3x = \delta(t) + e^{2t}u(t) + 2\delta(t-4)$ with rest IC.
(The $u(t)$ is there to make sure the input is 0 for $t < 0$.)

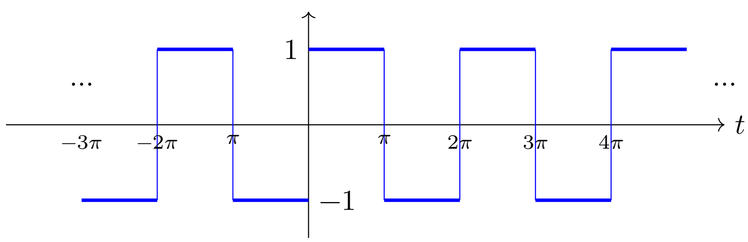
Problem 68. **(a)** Solve $2x'' + 8x' + 6x = \delta(t)$ with rest IC.

(b) Plug your solution into the DE and verify that it is correct

Problem 69. The graph of the function $f(t)$ is shown below. Compute the generalized derivative $f'(t)$. Identify the regular and singular parts of the derivative.



Problem 70. **Derivative of a square wave**
The graph below is of a function $\mathrm{sq}(t)$ (called a square wave). Compute and graph its generalized derivative.



Graph of $\mathrm{sq}(t)$ = square wave

Topic 21. Fourier series
Topic 22. Fourier series continuation

Problem 71. For each of the following:

(i) Find the Fourier series (no integrals needed)

(ii) Identify the fundamental frequency and corresponding base frequency.

(iii) Identify the Fourier coefficients a_n and b_n

(a) $\cos(2t)$

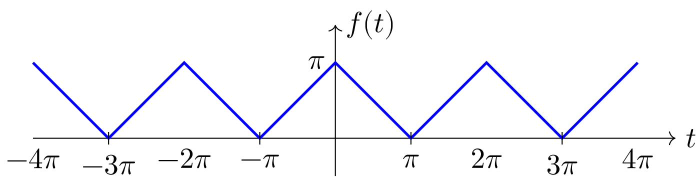
(b) $3\cos(2t - \pi/6)$

(c) $\cos(t) + 2\cos(5t)$

(d) $\cos(3t) + \cos(4t)$

Problem 72. Compute the Fourier series for the odd, period 2, amplitude 1 square wave. (Do this by computing integrals –not starting with the period 2π square wave.)

Problem 73. Compute the Fourier series for the period 2π triangle wave shown.



Topic 23. Fourier: calculation; sine and cosine series

Problem 74. Find the Fourier cosine series for the function $f(x) = x^2$ on $[0, 1]$. Graph the function and its even period 2 extension.

Problem 75. Find the Fourier series for the standard square wave shifted to the left so it's an even function, i.e., $sq(t + \pi/2)$.

Problem 76. Find the Fourier sine series for $f(t) = 30$ on $[0, \pi]$.

Topic 24. Fourier: ODEs

Problem 77. Solve $x' + kx = f(t)$, where $f(t)$ is the period 2π triangle wave with $f(t) = |t|$ on $[-\pi, \pi]$. (You can use the known series for $f(t)$.)

Problem 78. Solve $x'' + x' + 8x = g(t)$, where $g(t)$ is the period 2 triangle wave with $g(t) = |t|$ on $[-1, 1]$.

Problem 79. Solve $x'' + 16x = \displaystyle\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2(n^2-2)^2}$. Look out for resonance.

Topic 25. PDEs: separation of variables
Topic 26. Continuation

Problem 80. Let L=1. Solve the wave equation with boundary and initial conditions.

PDE: $y_{tt} = y_{xx}$

BC: $y(0,t) = 0$, $y(1,t) = 0$

IC: $y(x,0) = 30$, $y_t(x,0) = 0$.

Problem 81. Solve the heat equation with insulated ends.

(This problem uses a cosine series, so the $\lambda = 0$ case is important.)

(PDE) $u_t = 5u_{xx}$ for $0 \le x \le \pi$, $t > 0$.

(BC) $u_x(0,t) = 0$, $u_x(\pi,t) = 0$

(IC) $u(x,0) = x$.

Topic 27. Qualitative behavior of linear systems, phase portraits

Problem 82. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 83. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 84. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 85. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 86. Draw the trace-determinant diagram. Label all the parts with the type and dynamic stability of the critical point at the origin. Which types represent structurally stable systems?

(b) Give the equation for the parabola in the diagram. Explain where is comes from.

Problem 87. Consider the linear system $\mathbf{x}' = A\mathbf{x}$.

(a) Suppose A has $\mathrm{tr}(A) = -2.5$ and $\det(A) = 1$. Locate this system on the trace-determinant diagram. For this system, what is the type of the critical point at the origin?

(b) Compute the eigenvalues of this system and verify your answer in Part (a).

Problem 88. For each of the following linear systems, sketch phase portraits. Give the dynamic stability of the critical point at the origin. Give the structural stability of the system.

(a) $\mathbf{x}' = \begin{bmatrix} 5 & 1 \\ -4 & 10 \end{bmatrix}$

(b) $\mathbf{x}' = \begin{bmatrix} -7 & -3 \\ 3 & -17 \end{bmatrix}$

(c) $\mathbf{x}' = \begin{bmatrix} 5 & 3 \\ 0 & -2 \end{bmatrix}$

(d) $\mathbf{x}' = \begin{bmatrix} 5 & 5 \\ -5 & -1 \end{bmatrix}$

(e) $\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 4 & -3 \end{bmatrix}$

(f) $\mathbf{x}' = \begin{bmatrix} -4 & 4 \\ -1 & 0 \end{bmatrix} \mathbf{x}$

Topic 28. Qualitative behavior of non-linear systems
Topic 29. Structural stability

Problem 89. **(a)** Sketch the phase portrait for $x' = -x + xy, \quad y' = -2y + xy$.

(b) Consider x and y to be the sizes of two interacting populations. Tell a story about the populations.

Problem 90. Sketch the phase portrait for $x' = x^2 - y, \quad y' = x(1-y)$.

Draw one phase portrait for each possibility for the non-structurally stable critical point.

Problem 91. Consider the system: $x' = x - 2y + 3, \quad y' = x - y + 2$.

(a) Find the one critical point and linearize at it. For the linearized system, what is the type of the critical point?

(b) In Part (a) you should have found that the linearized system is a center. Since this is not structurally stable, it is not necessarily true that the nonlinear system has a center at the critical point. Nonetheless, in this case, it does turn out to be a nonlinear center. Prove this.

Problem 92. For the following system, draw the phase portrait by linearizing at the critical points.

$$x' = 1 - y^2, \quad y' = x + 2y.$$

Problem 93. For the following system, draw the phase portrait by linearizing at the critical points.

$$x' = x - y - x^2 + xy, \quad y' = -y - x^2.$$

Topic 30. Population models

Problem 94. Let $x(t)$ be the population of sharks off the coast of Massachusetts and $y(t)$ the population of fish. Assume that the populations satisfy the Volterra predator-prey equations

$$x' = ax - pxy; \qquad y' = -by + qxy, \qquad \text{where } a,\ b,\ p,\ q, \text{ are positive.}$$

Assume time is in years and a and b have units 1/years.

Suppose that, in a few years, warming waters start killing 10% of both the fish and the sharks each year. Show that the shark population will actually increase.

Problem 95. Consider the system of equations

$$x'(t) = 39x - 3x^2 - 3xy; \qquad y'(t) = 28y - y^2 - 4xy.$$

The four critical points of this system are (0,0), (13,0), (0,28), (5,8).

(a) Show that the linearized system at (0,0) has eigenvalues 39 and 28. What type of critical point is (0,0)?

(b) Linearize the system at (13,0); find the eigenvalues; give the type of the critical point.

(c) Repeat Part (b) for the critical point (0,28).

(d) Repeat Part (b) for the critical point (5,8).

(e) Sketch a phase portrait of the system. If this models two species, what is the relationship between the species? What happens in the long-run?

Problem 96. The system for this equation is

$$\begin{aligned} x' &= 4x - x^2 - xy \\ y' &= -y + xy \end{aligned}$$

(a) This models two populations with a predator-prey relationship. Which variable is the predator population?

(b) What would happen to the predator population in the absence of prey? What about the prey population in the absence of predators?

(c) There are three critical points. Find and classify them

(d) Sketch a phase portrait of this system. What is the relationship between the species? What happens in the long-run?

Problem 97. The equations for this system are

$$\begin{aligned} x' &= x^2 - 2x - xy \\ y' &= y^2 - 4y + xy \end{aligned}$$

(a) If this models two populations, what would happen to each of the populations in the absence of the other?

(b) There are four critical points. Find and classify them

(c) Sketch a phase portrait of the system. What is the relationship between the species? What happens in the long-run?

Topic 31. Physical models: the pendulum

We didn't cover physical models in class. This is still good practice for creating and interpreting phase portraits.

Problem 98. Nonlinear Spring
The following DE models a nonlinear spring:

$$m\ddot{x} = -kx + cx^3 \qquad \begin{cases} \textbf{hard if } c < 0 & \text{(cubic term adds to linear force)} \\ \textbf{soft if } c > 0 & \text{(cubic term opposes linear force).} \end{cases}$$

(a) Convert this to a companion system of first-order equations.

(b) Sketch a phase portrait of the system for both the hard and soft springs. You can use the fact that the linearized centers are also nonlinear centers. (This follows from energy considerations.)

(c) (Challenge! For anyone who is interested. This is not part of the ES.1803 syllabus.) Find equations for the trajectories of the system.

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