ES.1803 Linear Algebra Solutions, Spring 2024

Problem 1.

(a) *Find the eigenvalues and eigenvectors of* $A = \begin{bmatrix} 1 & -10 \\ 3 & -1 \end{bmatrix}$.

Solution: Characteristic equation: $\lambda^2 + 29 = 0 \quad \Rightarrow \lambda = \pm\sqrt{29}\, i$.

Basic eigenvectors (bases of Null$(A - \lambda I)$): We do this using the shortcut for 2×2 matrices instead of row reduction.

$\lambda = \sqrt{29}\, i$: $\quad (A - \lambda) = \begin{bmatrix} 1-\sqrt{29}i & -10 \\ 3 & -1-\sqrt{29}i \end{bmatrix}$. Basic eigenvector: $\mathbf{v} = \begin{bmatrix} 1+\sqrt{29}i \\ 3 \end{bmatrix}$. (Or we could take $\begin{bmatrix} 10 \\ 1-\sqrt{29}i \end{bmatrix}$.)

$\lambda = -\sqrt{29}i$: Use complex conjgate, basic eigenvector: $\overline{\mathbf{v_1}} = \begin{bmatrix} 1-\sqrt{29} \\ 3 \end{bmatrix}$.

(b) *Find the eigenvalues and eigenvectors of* $A = \begin{bmatrix} -8 & 7 \\ 1 & -2 \end{bmatrix}$.

Solution: Characteristic equation: $\lambda^2 + 10\lambda + 9 = 0 \quad \Rightarrow \lambda = -1, -9$.

Basic eigenvectors (bases of Null$(A - \lambda I)$): We do this using the shortcut for 2×2 matrices instead of row reduction.

$\lambda = -1$: $(A - \lambda I) = \begin{bmatrix} -7 & 7 \\ 1 & -1 \end{bmatrix}$. Basic eigenvector: $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda = -9$: $(A - \lambda I) = \begin{bmatrix} 1 & 7 \\ 1 & 7 \end{bmatrix}$. Basic eigenvector: $\mathbf{v} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$.

Problem 2.

Suppose that the matrix B has eigenvalues 2, 7 and 7, with eigenvectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

respectively.

(a) *Calculate* e^{Bt}.

Solution: S = matrix with eigenvectors as columns = $\begin{bmatrix} 1 & 1 & 5 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Λ = diagonal matrix of eigenvalues = $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

By diagonalization, $B = S\Lambda S^{-1}$. So,

$$e^{Bt} = Se^{\Lambda t}S^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 1 & 5 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{7t} & 0 \\ 0 & 0 & e^{7t} \end{bmatrix}\begin{bmatrix} 1 & -5 & 1 \\ 0 & 0 & 6 \\ 1 & 1 & -1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} e^{2t}+5e^{7t} & -5e^{2t}+5e^{7t} & -e^{2t}+e^{7t} \\ -e^{2t}+e^{7t} & 5e^{2t}+e^{7t} & e^{t}-e^{7t} \\ 0 & 0 & 6e^{7t} \end{bmatrix}.$$

(Really, the unmultiplied out form is the better answer!)

(b) *What are the eigenvalues and eigenvectors of e^{Bt}?*

Solution: Looking at the diagonalization $e^{Bt} = Se^{\Lambda t}S^{-1}$ it is clear the eigenvalues are e^t, e^{7t}, e^{7t} and the corresponding eigenvectors are the same as those of B.

(c) *Give an argument based on transformations why* $B = \begin{bmatrix} 1 & 1 & 5 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 5 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$ *has the eigenvalues and eigenvectors given in Part (a).*

Solution: We need to show $B \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $B \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $B \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$.

For writing ease, we'll write $B = S\Lambda S^{-1}$. The argument is the same for all three eigenvectors, so we'll just do the first one.

Since $S \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (first column of S), S^{-1} the opposite mapping: $S^{-1} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

We also know the diagonal matrix Λ maps $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to $2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

We can now see how $B = S\Lambda S^{-1}$ maps $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

$$S\Lambda S^{-1} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = S\Lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = S \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

This shows $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector of B with eigenvalue 2. The other eigenvalue/eigenvector pairs behave the same way.

(d) *What is the solution to $\mathbf{x}' = B\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$?*

Solution: Solution is

$$\mathbf{x}(t) = e^{Bt}\mathbf{x_0} = \frac{1}{6} \begin{bmatrix} e^t + 5e^{7t} & -5e^t + 5e^{7t} & -e^t + e^{7t} \\ -e^t + e^{7t} & 5e^t + e^{7t} & e^t - e^{7t} \\ 0 & 0 & 6e^{7t} \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} e^t + 5e^{7t} & -5e^t + 5e^{7t} \\ -e^t + e^{7t} & 5e^t + e^{7t} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \boxed{\frac{1}{3} \begin{bmatrix} e^t + 5e^{7t} \\ -e^t + e^{7t} \\ 0 \end{bmatrix}}.$$

(e) *Decouple the system $\mathbf{x}' = B\mathbf{x}$. That is, make a change of variables and write the DE in the new variables.*

Solution: Decoupling is just the change of variables $\mathbf{u} = S^{-1}\mathbf{x}$. So,

$$\mathbf{u} = S^{-1}\mathbf{x} \Leftrightarrow \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Leftrightarrow u = x/6 - 5y/6 + z/6;\ v = z;\ w = x/6 + y/6 - z/6.$$

In these coordinates the decoupled system is $\mathbf{u}' = \Lambda\mathbf{u}$:

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Problem 3.
Let $R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *and suppose R is the reduced row echelon form for A.*

(a) *What is the rank of A?*

Solution: A and R have the same rank. Two pivots in R implies rank $= 2$.

(b) *Find a basis for the null space of A.*

Solution: A and R have the same null space. The third and fourth variables are free. The strategy is to set one free variable to 1 and the others to 0 and solve for the pivot variables.

One way to do this computation is to write out the matrix multiplication as a linear combination of the columns and solve by inspection.

For example, set $x_3 = 1$, $x_4 = 0$ and solve for x_1 and x_2:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0} \quad \Rightarrow x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \mathbf{0}$$

By inspection we can see that $x_1 = -2,\ x_2 = -3$. So, one basis vector is $\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

Similarly, with $x_3 = 0$, $x_4 = 1$ we get a basis vector $\begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. Thus a basis is $\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

One simple way to present this is to show all the work underneath the columns, i.e.,

$$\begin{array}{cccc} \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \begin{matrix} 2 \\ 3 \\ 0 \end{matrix} & \left.\begin{matrix} 3 \\ 1 \\ 0 \end{matrix}\right] \\ x_1 & x_2 & x_3 & x_4 \\ -2 & -3 & 1 & 0 \\ -3 & -1 & 0 & 1 \end{array}$$

(c) *Suppose the column space of A has basis $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}3\\1\\1\end{bmatrix}$. Find a possible matrix for A. That is, give a matrix with RREF R and the given column space.*

Solution: Looking at R, Columns 1 and 2 are pivot columns. We put the given basis vectors in those columns:

$$A = \begin{bmatrix}1 & 3 & * & *\\1 & 1 & * & *\\0 & 1 & * & *\end{bmatrix}$$

The free columns of R are linear combinations of the pivot columns and those of A are the same linear combinations. In R it is clear that

$$\text{Col}_3 = 2\cdot\text{Col}_1 + 3\cdot\text{Col}_2 \quad \text{and} \quad \text{Col}_4 = 3\cdot\text{Col}_1 + \text{Col}_2.$$

So,

$$A = \begin{bmatrix}1 & 3 & 11 & 6\\1 & 1 & 5 & 4\\0 & 1 & 3 & 1\end{bmatrix}.$$

(d) *Find a matrix with the same reduced echelon form but such that $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\\3\end{bmatrix}$ are in its column space.*

Solution: We found the relationships between the columns in in Part (c). So we put the given columns as pivot columns and construct the free columns from these relationships:
$\begin{bmatrix}1 & 1 & 5 & 4\\1 & 2 & 8 & 5\\1 & 3 & 11 & 6\end{bmatrix}$

Note: you could put any other basis for the subspace generated by $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\\3\end{bmatrix}$ in the pivot columns and adjust the free columns accordingly.

Problem 4.

Suppose $A = \begin{bmatrix}a & b & c\\0 & 2 & e\\0 & 0 & 3\end{bmatrix}$.

(a) *What are the eigenvalues of A?*

Solution: For an upper triangular matrix the eigenvalues are the diagonal entries: a, 2, 3.

(b) *For what value (or values) of a, b, c, e is A singular (non-invertible)?*

Solution: $\det(A)$ = product of eigenvalues. So A is singular when $a = 0$. The parameters b, c, e can take any values.

(c) *What is the minimum rank of A (as a, b, c, e vary)? What's the maximum?*

Solution: When $a = 0$, the null space is dimension 1, so rank $=2$.

When $a \neq 0$, A is invertible, so has rank $= 3$.

(d) *Suppose $a = -5$. In the system $\mathbf{x}' = A\mathbf{x}$, is the equilibrium at the origin stable or unstable.*

Solution: The two positive eigenvalues imply the system is unstable.

Problem 5.

Suppose that $A = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1}$.

(a) *What are the eigenvalues of A?*

Solution: The eigenvalues are the same as the diagonal matrix, i.e., 1, 2, 3.

(b) *Express A^2 and A^{-1} in terms of S.*

Solution: $A^2 = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} S^{-1}; \quad A^{-1} = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} S^{-1}.$

(c) *What would I need to know about S in order to write down the most rapidly growing exponential solution to $\mathbf{x}' = A\mathbf{x}$?*

Solution: You need to know the eigenvector that goes with the eigenvalue 3. That is, you need to know the third column of S.

Problem 6.

(a) *An orthogonal matrix is one where the columns are orthonormal (mutually orthogonal and unit length). Equivalently, S is orthogonal if $S^{-1} = S^T$.*

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. *Find an orthogonal matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$*

Solution: The problem is asking us to diagonalize A, taking care that the matrix S is orthogonal.

A has characteristic equation: $\lambda^2 - 2\lambda - 3$. So it has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$. By inspection (or computation), we have eigenvectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

These are clearly orthogonal to each other. We normalize their lengths and use the normalized eigenvectors in the matrix S.

$$S = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}; \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \quad \Rightarrow A = S\Lambda S^{-1}.$$

Note: A is a symmetric matrix. It turns out that symmetric matrix has an orthonormal set of basic eigenvectors.

(b) *Decouple the equation $\mathbf{x}' = A\mathbf{x}$, with* $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Solution: The decoupling change of variable is $\mathbf{u} = S^{-1}\mathbf{x} \Leftrightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The decoupled system is $\mathbf{u}' = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{u} \Leftrightarrow \begin{cases} u_1' = -u_1 \\ u_2' = 3u_2 \end{cases}$.

Problem 7.

Suppose A has eigenvalues -2 and -3 with corresponding eigenvectors $\begin{bmatrix}1\\2\end{bmatrix}$ *and* $\begin{bmatrix}1\\-1\end{bmatrix}$.

(a) *Compute A^{-1} explicitly.*

Solution: We know $A = S\Lambda S^{-1}$, where $S = \begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}$ is the matrix of eigenvectors and $\Lambda = \begin{bmatrix}-2 & 0\\0 & -3\end{bmatrix}$. So, $A^{-1} = S\Lambda^{-1}S^{-1}$. Computing, we find $S^{-1} = \frac{1}{3}\begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}$ and

$$A^{-1} = S\Lambda^{-1}S^{-1} = \tfrac{1}{3}\begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}-1/2 & 0\\0 & -1/3\end{bmatrix}\begin{bmatrix}1 & 1\\2 & -1\end{bmatrix} = \boxed{\frac{1}{18}\begin{bmatrix}-7 & -1\\-2 & -8\end{bmatrix}}.$$

(b) *Consider the system* $\begin{bmatrix}x'\\y'\end{bmatrix} = A\begin{bmatrix}x\\y\end{bmatrix}$. *Find a change of coordinates*

$$u = ax + by, \qquad v = cx + dy$$

so that in these new coordinates the system becomes $u' = r_1 u$ and $v' = r_2 v$. Also give the values of r_1 and r_2.

Solution: $\begin{bmatrix}u\\v\end{bmatrix} = S^{-1}\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$.

Thus, $\boxed{u = x/3 + y/3; \quad v = 2x/3 - y/3}$, r_1 and r_2 are the eigenvalues $\boxed{r_1 = -2,\ r_2 = -3.}$

Problem 8.

Let $A = \begin{bmatrix}1 & 4 & 2 & 2\\2 & 8 & 1 & 9\\1 & 4 & 1 & 7\end{bmatrix}$

(a) *Put A in reduced row echelon form.*

Solution:

$$\begin{bmatrix}1 & 4 & 2 & 2\\2 & 8 & 5 & 9\\1 & 4 & 3 & 7\end{bmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix}1 & 4 & 2 & 2\\0 & 0 & 1 & 5\\0 & 0 & 1 & 5\end{bmatrix} \xrightarrow{R_3 = R_3 - R_1} \begin{bmatrix}1 & 4 & 2 & 2\\0 & 0 & 1 & 5\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\xrightarrow{R_1 = R_2 - 2R_2} \boxed{\begin{bmatrix}1 & 4 & 0 & -8\\0 & 0 & 1 & 5\\0 & 0 & 0 & 0\end{bmatrix}}.$$

(b) *Give a basis for the column space of A.*

Solution: The pivot columns are Columns 1 and 3. These columns of A give a basis for the column space: $\boxed{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}.}$

Problem 9.

The matrix A has reduced row echelon form $R = \begin{bmatrix}1 & 5 & 0 & 4\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\end{bmatrix}$.

(a) *What is the rank of A?*

Solution: Two pivots, so rank = 2.

(b) *Find a basis for the null space of A.*

Solution: The reduced matrix and A have the same null space. The free variables are x_2 and x_4. Using our usual notation to set them alternately to 1 and 0, we find

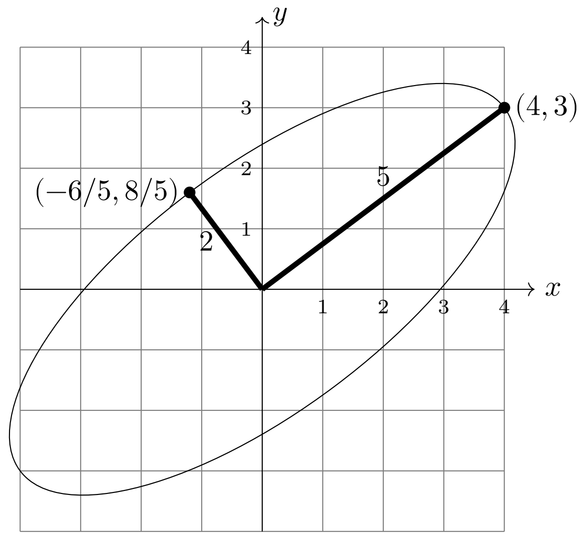
$$\begin{array}{cccc} \left[\begin{matrix} 1 & 5 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{matrix}\right] \\ x_1 & x_2 & x_3 & x_4 \\ -5 & 1 & 0 & 0 \\ -4 & 0 & -2 & 1 \end{array}$$

The vectors $\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ form a basis of the null space.

(c) *Find a matrix A with reduced row echelon form R and such that the equations $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ can both be solved.*

Solution: We can only solve $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b}$ is in the column space. So we put the two vectors above as the pivot columns of A $\begin{bmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \\ 1 & * & 0 & * \end{bmatrix}$. The free columns must satisfy the same relations with the pivot columns as R. These were found when we found the null space. That is, $\text{Col}_2 = 5\,\text{Col}_1$ and $\text{Col}_3 = 4\,\text{Col}_1 + 2\,\text{Col}_3$. We get $\boxed{A = \begin{bmatrix} 1 & 5 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 1 & 5 & 0 & 4 \end{bmatrix}}$.

Problem 10. (a) *Consider the ellipse shown. The axes are drawn in with their lengths and endpoints.*



Find a matrix A such that multiplication by A transforms this ellipse into the unit circle.

Solution: We can write A in one go by noting that we want A to map the ellipse's major axis to $\begin{bmatrix}1 & 0\end{bmatrix}^T$ and its minor axis to $\begin{bmatrix}0, 1\end{bmatrix}^T$. We are given these axes, so we can easily write the inverse matrix that sends the standard basis to the axes of the ellipse.
$A^{-1} = \begin{bmatrix}4 & -6/5 \\ 3 & 8/5\end{bmatrix} \Rightarrow \boxed{A = \frac{1}{10}\begin{bmatrix}8/5 & 6/5 \\ -3 & 4\end{bmatrix} = \begin{bmatrix}8/50 & 6/50 \\ -3/10 & 4/10\end{bmatrix}.}$

(You could also find this as a rotation by $-\tan^{-1}(3/4)$ followed by scaling x by $1/5$ and y by $1/2$.)

(b) *Suppose A is a matrix with eigenvalue λ and corresponding eigenvector $\mathbf{v}$. Show that the block matrix $\begin{bmatrix}0 & A \\ A & 0\end{bmatrix}$ has eigenvalues $\pm\lambda$ and find an eigenvector for each one.*

Solution: Since $A\mathbf{v} = \lambda\mathbf{v}$, we have $\begin{bmatrix}0 & A \\ A & 0\end{bmatrix}\begin{bmatrix}\mathbf{v} \\ \mathbf{v}\end{bmatrix} = \begin{bmatrix}A\mathbf{v} \\ A\mathbf{v}\end{bmatrix} = \begin{bmatrix}\lambda\mathbf{v} \\ \lambda\mathbf{v}\end{bmatrix}$. So, $\begin{bmatrix}\mathbf{v} \\ \mathbf{v}\end{bmatrix}$ is an eigenvector with eigenvalue λ.

Likewise, $\begin{bmatrix}0 & A \\ A & 0\end{bmatrix}\begin{bmatrix}\mathbf{v} \\ -\mathbf{v}\end{bmatrix} = \begin{bmatrix}-A\mathbf{v} \\ A\mathbf{v}\end{bmatrix} = \begin{bmatrix}-\lambda\mathbf{v} \\ \lambda\mathbf{v}\end{bmatrix} = -\lambda\begin{bmatrix}\mathbf{v} \\ -\mathbf{v}\end{bmatrix}$. We've found an eigenvector with eigenvalue $-\lambda$.

End linear algebra practice for final solutions