ES.1803 Linear Algebra Practice, Spring 2024

Problem 1.

(a) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -10 \\ 3 & -1 \end{bmatrix}$.

(b) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -8 & 7 \\ 1 & -2 \end{bmatrix}$.

Problem 2.

Suppose that the matrix B has eigenvalues 2, 7 and 7, with eigenvectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

respectively.

(a) Calculate e^{Bt}.

(b) What are the eigenvalues and eigenvectors of e^{Bt}?

(c) Give an argument based on transformations why $B = \begin{bmatrix} 1 & 1 & 5 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 5 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$ has the eigenvalues and eigenvectors given in Part (a).

(d) What is the solution to $\mathbf{x}' = B\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$?

(e) Decouple the system $\mathbf{x}' = B\mathbf{x}$. That is, make a change of variables and write the DE in the new variables.

Problem 3.

Let $R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and suppose R is the reduced row echelon form for A.

(a) What is the rank of A?

(b) Find a basis for the null space of A.

(c) Suppose the column space of A has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Find a possible matrix for A. That is, give a matrix with RREF R and the given column space.

(d) Find a matrix with the same reduced echelon form but such that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are in its column space.

Problem 4.

Suppose $A = \begin{bmatrix} a & b & c \\ 0 & 2 & e \\ 0 & 0 & 3 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) For what value (or values) of a, b, c, e is A singular (non-invertible)?

(c) What is the minimum rank of A (as a, b, c, e vary)? What's the maximum?

(d) Suppose $a = -5$. In the system $\mathbf{x}' = A\mathbf{x}$, is the equilibrium at the origin stable or unstable.

Problem 5.

Suppose that $A = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1}$.

(a) What are the eigenvalues of A?

(b) Express A^2 and A^{-1} in terms of S.

(c) What would I need to know about S in order to write down the most rapidly growing exponential solution to $\mathbf{x}' = A\mathbf{x}$?

Problem 6.

(a) An orthogonal matrix is one where the columns are orthonormal (mutually orthogonal and unit length). Equivalently, S is orthogonal if $S^{-1} = S^T$.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find an orthogonal matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$

(b) Decouple the equation $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Problem 7.

Suppose A has eigenvalues -2 and -3 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) Compute A^{-1} explicitly.

(b) Consider the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$. Find a change of coordinates

$$u = ax + by, \qquad v = cx + dy$$

so that in these new coordinates the system becomes $u' = r_1 u$ and $v' = r_2 v$. Also give the values of r_1 and r_2.

Problem 8.

Let $A = \begin{bmatrix} 1 & 4 & 2 & 2 \\ 2 & 8 & 1 & 9 \\ 1 & 4 & 1 & 7 \end{bmatrix}$

(a) Put A in reduced row echelon form.

(b) Give a basis for the column space of A.

Problem 9.

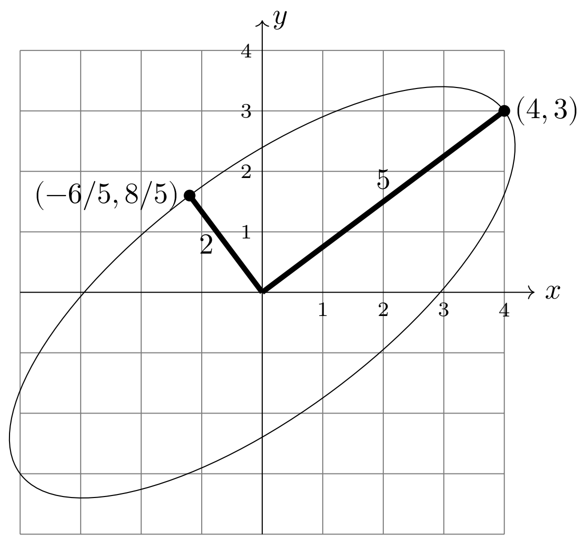
The matrix A has reduced row echelon form $R = \begin{bmatrix} 1 & 5 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) What is the rank of A?

(b) Find a basis for the null space of A.

(c) Find a matrix A with reduced row echelon form R and such that the equations $A\mathbf{x} = \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $A\mathbf{x} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ can both be solved.

Problem 10. (a) Consider the ellipse shown. The axes are drawn in with their lengths and endpoints.



Find a matrix A such that multiplication by A transforms this ellipse into the unit circle.

(b) Suppose A is a matrix with eigenvalue λ and corresponding eigenvector $\mathbf{v}$. Show that the block matrix $\begin{bmatrix}0 & A\\A & 0\end{bmatrix}$ has eigenvalues $\pm\lambda$ and find an eigenvector for each one.

End linear algebra practice for final