ES.1803 Practice Solutions – Final Quiz, Spring 2024

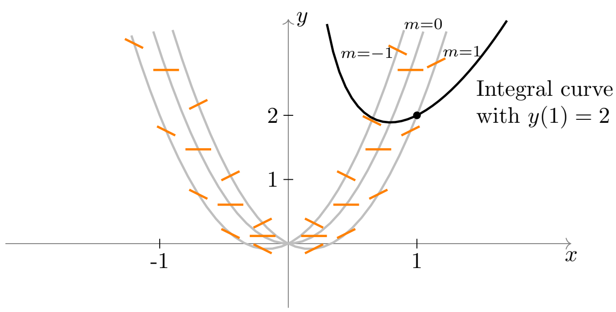
Important: Not every topic is covered here. When preparing for the final be sure to look over other review materials as well as old psets and exams.

Problem 1. *For the DE* $\dfrac{dy}{dx} = -\dfrac{y}{x} + 3\,x$*:*

(a) *Sketch the direction field for this DE, using (light or dotted) isoclines for the slopes -1 and 0.*

Solution: See picture below. Isoclines: $y' = -\frac{y}{x} + 3x = m \Rightarrow y = 3x^2 - mx$.

(Note problem at $(0, 0)$.)



(b) *For the solution curve passing through the point (1,2): If Euler's method with step-size $h = 0.1$ were used to approximate $y(1.1)$, would the approximation come out too high or too low? Explain.*

Solution: If $y(1) = 2$ then $y'(1) = -\frac{2}{1} + 3 \cdot 1 = 1$.

$y''(1) = -\frac{xy'-y}{x^2} + 3$, so $y''(1) = 4 > 0$. This shows the integral curve is concave up at $(1, 2)$, which implies the estimate is probably too low.

(c) *For the solution with $y(1) = 2$, compute the Euler approximation to $y(1.1)$ using step-size $h = 0.1$.*

Solution: This takes only one step, so we don't bother with a table. Euler:

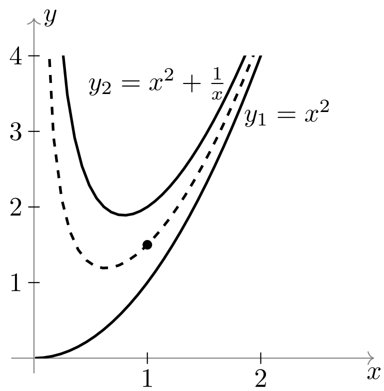
$$y_1 = y_0 + hf(x_0, y_0) = 2 + 0.1 \cdot f(1, 2) = 2.1 \quad \Rightarrow y(1.1) \approx 2.1.$$

(d) *The functions $y_1 = x^2$ and $y_2 = x^2 + \frac{1}{x}$ are solutions to this DE. If $y = y(x)$ is the solution satisfying the IC $y(1) = 1.5$, show that $100 \le y(10) \le 100.1$. Do we need to include the equal signs in this inequality? Why or why not?*

Solution: The picture below shows the plots of the 2 given solutions.

The dotted line indicates that (by the Existence and uniqueness theorem) the solution with IC (1,1.5) must stay between these two solutions. So, $y_1(10) < y(10) < y_2(10)$. Thus, $100 < y(10) < 100.1$.

Since the solutions can't touch, we don't need the equal signs.



(e) *Find the general solution the DE and verify the prediction of Part (b).*

Solution: This equation is first-order linear, so we can use the variation of parameters formula to solve it. In standard form the DE is $y' + \dfrac{1}{x}\,y = 3x$.

Homogeneous solution: $\quad y_h(x) = \int e^{-1/x}\,dx = 1/x.$

Variation of parameters: $\quad y(x) = y_h(x)\int 3x^2\,dx + cy_h(x) = \boxed{x^2 + c/x.}$

(An even quicker method would be to use the linearity and the two solutions given in Part (d) to get the general homogeneous solution: $y_h = c(y_1 - y_2) = c/x$.

Then, the general solution is $y(x) = x^2 + c/x$. (particular plus homeogeneous.)

Problem 2. *Let* $P(D) = D^2 + bD + 5I$ *where* $D = \frac{d}{dt}$.

(a) *For what range of the values of* $b \geq 0$ *will the solutions to* $P(D)y = 0$ *exhibit oscillatory behavior?*

Solution: Characteristic polynomial: $P(r) = r^2 + br + 5$. Roots: $r = \dfrac{-b \pm \sqrt{b^2 - 20}}{2}$.

Solutions are oscillatory when r is complex. i.e., if $b^2 - 20 < 0 \Rightarrow b < \sqrt{20} = 2\sqrt{5}$.

(b) *For* $b = 4$, *solve the DEs* *(i)* $P(D)y = 4\,e^{2t}\sin(t)$ *(ii)* $P(D)y = 4\,e^{2t}\cos(t)$

Write your answers in both amplitude-phase and rectangular form.

Solution: (i) We'll complexify and use the exponential response formula.

Complexify: $P(D)z = e^{(2+i)t}$, where $y = \operatorname{Im}(z)$

ERF: $z_p(t) = 4\,\frac{e^{(2+i)t}}{P(2+i)}$.

Calculate: $P(2+i) = (2+i)^2 + 4(2+i) + 5 = 16 + 8i = 8(2+i)$

$$P(2+i) = 8(2+i), \quad |P(2+i)| = 8\sqrt{5}, \quad \boxed{\phi = \operatorname{Arg}(P(2+i)) = \tan^{-1}(1/2) \text{ in Q1}}.$$

So, $z_p(t) = \dfrac{4e^{2t}e^{i(t-\phi)}}{8\sqrt{5}} = \dfrac{e^{2t}e^{i(t-\phi)}}{2\sqrt{5}} \quad \Rightarrow \boxed{y_p(t) = \operatorname{Im}(z_p) = \dfrac{e^{2t}}{2\sqrt{5}}\sin(t-\phi)}.$

(Rectangular form is below.)

(ii) We can reuse z_p from Part (i): $\boxed{y_p(t) = \text{Re}(z_p) = \dfrac{e^{2t}}{2\sqrt{5}}\cos(t-\phi)}$.

The problem also asks for the solution in rectangular form. We can find that from the polar form using trig identities or directly from the complexified solution. We'll do the latter:

$$z_p(t) = \frac{e^{2t}(\cos(t) + i\sin(t))}{8(2+i)}\frac{(2-i)}{(2-i)} = \frac{e^{2t}}{10}[(2\cos(t) + \sin(t)) + i(2\sin(t) - \cos(t))]$$

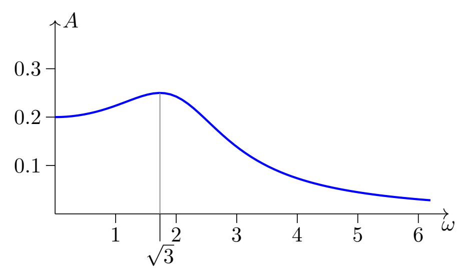
Thus, $\boxed{\text{(i) } y_p(t) = \text{Im}(z_p) = \dfrac{e^{2t}(2\sin(t) - \cos(t))}{10}, \quad \text{(ii) } y_p(t) = \text{Re}(z_p) = \dfrac{e^{2t}(2\cos(t) + \sin(t))}{10}.}$

(c) *Given $b = 2$, for what ω does $P(D)y = \cos(\omega t)$ have the biggest response?*

Solution: The sinusoidal resonse formula gives: $y_p(t) = \dfrac{\cos(\omega t - \phi)}{|P(i\omega)|}$

So the amplitude of the response is $A(\omega) = \dfrac{1}{|P(i\omega)|} = \dfrac{1}{\sqrt{(5-\omega^2)^2 + 4\omega^2}}$

$A(w)$ has a maximum when $(5-\omega^2)^2 + 4\omega^2$ has a minimum. Using simple calculus we find the maximum response is at $\boxed{\omega = \sqrt{3}.}$



Amplitude of output

Problem 3. *Find the general solution to the DE $(D^3 - I)\,y = e^x$*

Express the answer using real-valued functions only.

Solution: Characteristic polynomial: $P(r) = r^3 - 1$.

Roots: r = cube roots of unity $= 1,\ e^{2\pi i/3},\ e^{4\pi i/3} = 1,\ -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

General (real) homogeneous solution: $\boxed{y_h(x) = c_1 e^x + c_2\, e^{-x/2}\cos\left(\dfrac{\sqrt{3}}{2}\,x\right) + c_3\, e^{-x/2}\sin\left(\dfrac{\sqrt{3}}{2}\,x\right).}$

Particular solution: $P(1) = 0,\ P'(1) = 3 \neq 0$. So, $\boxed{y_p(x) = \dfrac{xe^x}{P'(1)} = \dfrac{1}{3}xe^x.}$

General solution: $\boxed{y(x) = y_p(x) + y_h(x).}$

Problem 4. *Let L denote the differential operator $Ly = D^2 y - \frac{1}{x}\,Dy + 4x^2 y$, where $D = \frac{d}{dx}$.*

(a) *Show that the DE $Ly = 0$ has solutions $y_1(x) = \cos(x^2)$ and $y_2(x) = \sin(x^2)$.*

Solution: Direct calculation shows $Ly_1 = Ly_2 = 0$.

(b) *Show that the initial value problem*

$$Ly = 0, \quad y(0) = 0, \; y'(0) = 0$$

has more than one solution. Why doesn't this contradict the Existence and Uniqueness Theorem? On what intervals does existence and uniqueness hold?

Solution: For any c_2 the function $y(x) = c_2 \sin(x^2)$ solves the IVP.

This doesn't violate existence and uniqueness because the coefficient $p(x) = \frac{1}{x}$ is not continuous at 0.

Existence and uniqueness holds where the coefficients are continuous, i.e., on $(0, \infty)$ and $(-\infty, 0)$.

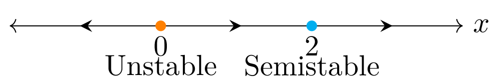
Problem 5. *Suppose that a population of variable size (in some suitable units) $p(t)$ follows the growth law* $\dfrac{dp}{dt} = p^3 - 4p^2 + 4p$. *Without solving the DE explicitly:*

(a) *Find all critical points and classify each according to its stability type using a phase line diagram.*

Solution: $p^3 - 4p^2 + 4p = p(p-2)^2 \Rightarrow$ critical points are $p = 0$ and $p = 2$.

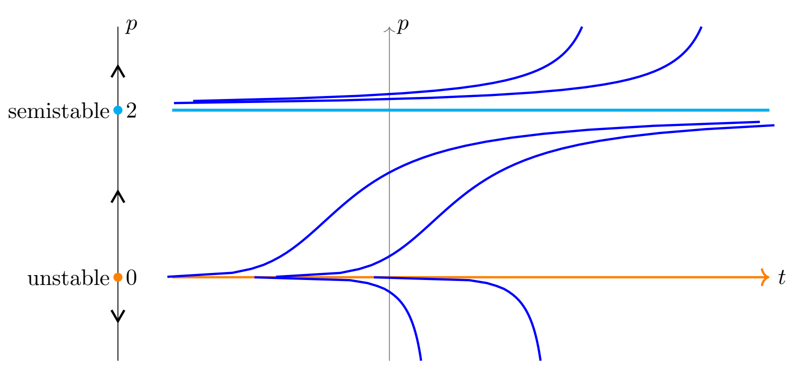
By looking at the phase line we see that 0 is unstable and 2 is semi-stable.

Here is the phase line (drawn horizontally to save space.)



(b) *Draw a rough sketch (on p-vs.-t axes) of the family of solutions. What happens to the population in the long-run if it starts out at size 1 unit; at size 3 units ?*

Solution: By hand a rough sketch is fairly easy: First draw horizontal lines at $p = 0$ and $p = 2$ (the equilibrium solutions). Next draw the solutions above $p = 2$ as curves curving upward, those between 0 and 2 are 'logistic-like' going from 0 to 2 and those below 0 curve down. Below is a plot of the solutions. In the long run if $p(0) = 1$ then $p \to 2$ and if $p(0) = 3$ then $p \to \infty$.



(c) *Explain why the rate equation given by the DE was all we needed to get the answer to Part (b).*

Solution: The rate information was all we needed to draw the phase line and determine the critical points and their stability types.

(d) *Now we'll add a harvesting parameter to the system:* $p' = p^3 - 4p^2 + 4p - r$.

(i) Draw the bifurcation diagram for this system.

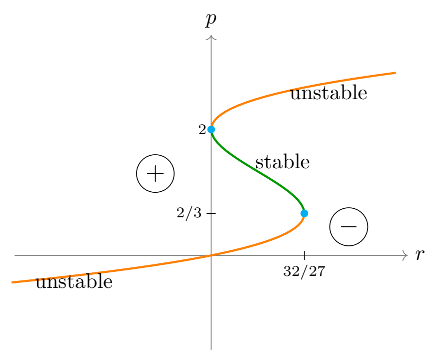
(ii) Give the bifurcation points.

(iii) For what values of r is the population sustainable?

Solution: (i) The critical points are $p' = p^3 - 4p^2 + 4p - r = 0 \quad \Rightarrow r = p^3 - 4p^2 + 4p$.

To plot this, first note that $r = p(p-2)^2$. So, $r = 0$ when $p = 0$ or $p = 2$. With calculus, we can find the maxima and minima for r. The only minumum occurs when $p = 2$. The only maximum is when $p = 2/3$. (So, $r = 32/27$.) This helps us plot the cubic. Since r is the horizontal axis, the cubic is turned sideways.

Once we have the plot, we can add pluses and minuses indicating the regions in the rp-plane where p' is positive or negative. From this we can label the parts of the bifurcation diagram as stable or unstable.



(ii) There are bifurcation points at $r = 0$ and $r = 32/27$.

(iii) Sustainability requires a positive stable critical point. So, this system is sustainable for $0 < r < 32/27$.

Problem 6. (a) *Solve* $2y'' - 2y' - 4y = \delta(t)$, *with rest IC.*

Solution: The characteristic roots are $2, -1$. So,the general homogeneous solution is $y(t) = c_1 e^{2t} + c_2 e^{-t}$.

Rest IC means the pre-intial conditions are $y(0^-) = 0$, $y'(0^-) = 0$.

The impulse at $t = 0$ divides the problem into cases.

<u>Case $t < 0$</u>. On this interval, the DE and initial conditions are

$$2y'' - 2y' - 4y = 0, \quad y(0^-) = 0,\ y'(0^-) = 0.$$

Clearly $y(t) = 0$, is the solution on $t < 0$.

<u>Case $t > 0$</u>. The delta function input causes a jump at $t = 0$. The post-initial conditions are $y(0^+) = y(0^-) = 0$, $y'(0^+) = y'(0^-) + 1/2 = 1/2$. Thus, on this interval, the DE and initial conditions are

$$2y'' - 2y' - 4y = 0, \quad y(0^+) = 0,\ y'(0^+) = 1/2.$$

This has solution, $y(t) = c_1 e^{2t} + c_2 e^{-t}$. At $t = 0^+$, we get

$$y(0^+) = c_1 + c_2 = 0 \quad \text{and} \quad y'(0^+) = 2c_1 - c_2 = 1/2.$$

Solving these equations we find: $c_1 = 1/6$, $c_2 = -1/6$. Therefore, the full solution is

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{6}e^{2t} - \frac{1}{6}e^{-t} & \text{for } t > 0. \end{cases}$$

(b) *Solve $2y'' + 2y = \delta(t-3)$, with rest IC.*

Solution: The characteristic roots are $\pm i$. So,the general homogeneous solution is $y(t) = c_1 \cos(t) + c_2 \sin(t)$.

Rest IC means the pre-intial conditions are $y(0^-) = 0$, $y'(0^-) = 0$.

The impulse at $t = 3$ divides the problem into cases.

Case $t < 3$. On this interval, the DE and initial conditions are

$$2y'' + 2y = 0, \quad y(0^-) = 0,\ y'(0^-) = 0.$$

Clearly $y(t) = 0$, is the solution on $t < 3$.

Looking ahead to the next case, we have $y(3^-) = 0$, $\quad y'(3^-) = 0$.

Case $t > 3$. The impulse cause a jump in velocity at $t = 3$. The post-initial conditions are $y(3^+) = y(3^-) = 0$, $y'(3^+) = y'(3^-) + 1/2 = 1/2$. Thus, on this interval, the DE and initial conditions are

$$2y'' + 2y = 0, \quad y(3^+) = 0,\ y'(3^+) = 1/2.$$

Because the system is time invariant, we can write the general homogeneous solution as

$$y(t) = c_1 \cos(t-3) + c_2 \sin(t-3).$$

Now we use the post-IC to find c_1 and c_2. We get $c_1 = 0$, $c_2 = 1/2$. Therefore, the full solution is

$$y(t) = \begin{cases} 0 & \text{for } t < 3 \\ \frac{1}{2}\sin(t-3) & \text{for } t > 3. \end{cases}$$

(c) *Solve $x' + tx = \delta(t-5)$*

Solution: Using the variation of parameters formula: $x_h(t) = e^{-\frac{t^2}{2}}$ so

$$\begin{aligned} x(t) &= e^{-\frac{t^2}{2}} \left[\int \delta(t-5) e^{\frac{t^2}{2}}\, dt + c \right] \\ &= e^{-\frac{t^2}{2}} e^{\frac{25}{2}} u(t-5) + c e^{-\frac{t^2}{2}} \end{aligned}$$

Here we used that $\displaystyle\int \delta(t-a)\, dt = u(t-a)$ and that $e^{\frac{t^2}{2}}\delta(t-5) = e^{\frac{5^2}{2}}\delta(t-5)$.

Problem 7. *Let $f(t)$ be 0 for $t < 0$ and $3e^{-2t}$ for $t > 0$. Compute the generalized derivative of $f(t)$.*

Solution: The function has a jump of 3 at the origin. This adds a singular part to the generalized derivative: $f'(t) = 3\delta(t) + \begin{cases} 0 & \text{for } t < 0 \\ -6e^{-2t} & \text{for } t > 0 \end{cases}$

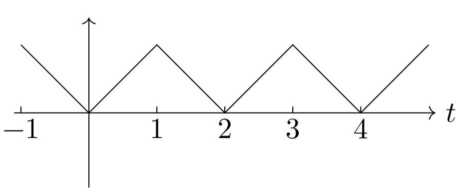
Problem 8. *For* $f(t) = t$ *on* $0 < t < 1$:

(a) *Sketch the following periodic extensions of f over three or more full periods in the cases.*

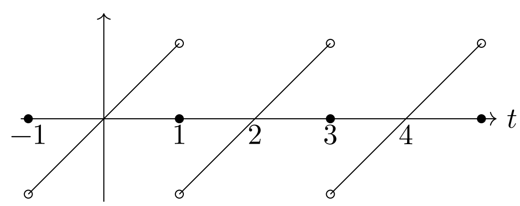
(i) Even period 2 extension (ii) Odd period 2 extension (iii) Period 1 extension.

In all three cases chose endpoint values that show where the Fourier series expansion will converge. (Do this without *computing the Fourier series).*

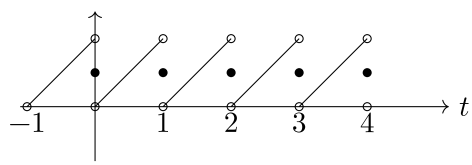
Solution: The Fourier series will converge to f except at the points of discontinuity, where it will converge to the midpoint of the jump. Thus cases (ii) and (iii) require some care with the endpoints. Here are the extensions.



(i) Even period 2 extension



(ii) Odd period 2 extension



(iii) Period 1 extension

(b) *Compute the Fourier sine series of f.*

Solution: $b_n = 2\int_0^1 t\sin(n\pi t)\,dt = 2\left(-\frac{t\cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{n^2\pi^2}\bigg|_0^1\right) = -2\frac{(-1)^n}{n\pi}$

Thus, on $[0, 1]$, we have $\boxed{f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi t).}$

(c) *Find the periodic solution to the DE* $x'' + 10x = \tilde{f}_{odd}(t)$. *Does near-resonance occur in this situation? If so, which frequency in the 'driving force' $\tilde{f}_{odd}(t)$ produces it?*

Solution: The characteristic polynomial is $P(r) = r^2 + 10$. So,

$$P(in\pi) = 10 - (n\pi)^2 \quad \Rightarrow |P(in\pi)| = |10 - n^2\pi^2|, \boxed{\phi(n) = \text{Arg}(P(in\pi)) = \begin{cases} 0 & \text{if } n = 1 \\ \pi & \text{if } n \geq 2. \end{cases}}$$

(Note: $10 - n^2\pi^2 \neq 0$ for any integer n).

Using the SRF: The periodic solution to $x_n'' + 10x_n = \sin(n\pi t)$ is

$$x_{n,p}(t) = \frac{\sin(n\pi t - \phi(n))}{|P(in\pi)|} = \frac{\sin(n\pi t - \phi(n))}{|10 - n^2\pi^2|}.$$

Superposition (using the infinite Fourier series) gives the periodic solution to the DE:

$$\boxed{x_p(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n|10-n^2\pi^2|}\sin(n\pi t - \phi(n)) = \frac{2}{\pi}\frac{\sin(\pi t)}{|10-\pi^2|} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n|10-n^2\pi^2|}\sin(n\pi t)}$$

The last expression was found by using the known values of $\phi(n)$.

Since $\pi^2 \approx 9.87 \approx 10$ the $n = 1$ term produces the largest response (i.e., 'near resonance').

(d) *Solve the DE* $x' + 10x = \tilde{f}_{odd}(t)$.

Solution: Note the differential operator is different than in Part (c).

As in Part (c), once we solve $x_n' + 10x_n = \sin(n\pi t)$ we can use superposition.

In preparation for using the SRF we compute

$$P(in\pi) = 10+in\pi; \quad |P(in\pi)| = \sqrt{10^2+n^2\pi^2}; \quad \boxed{\phi(n) = \mathrm{Arg}(P(in\pi)) = \tan^{-1}(n\pi/10) \text{ in Q1}}.$$

Now the SRF gives

$$x_{n,p}(t) = \frac{\sin(n\pi t - \phi(n))}{|P(in\pi)|} = \frac{\sin(n\pi t - \phi(n))}{\sqrt{10^2+n^2\pi^2}}$$

Using superposition:

$$\boxed{x_p(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(n\pi t - \phi(n))}{n\sqrt{10^2+n^2\pi^2}}.}$$

As a reward for dealing with a first-order system we know there can't be near-resonance because the $100 + n^2\pi^2$ term in the denominator is never small.

Problem 9. (a) *Write down the wave equation with IC's and BC's for the string of length 1, with clamped ends, wave speed 2, initially at equilibrium, struck at time 0. Then derive the Fourier series solution using separation of variables.*

Solution: PDE: $y_{tt} = 4y_{xx}$

BC: $y(0,t) = y(1,t) = 0$

IC: $y(x,0) = 0$ (initially at equilibrium)
$y_t(x,0) = f(x)$ (initial velocity right after being struck)

We solve using the Fourier method of separation of variables.

Step 1. Find separated solutions to the PDE: $y(x,t) = X(x)T(t)$

Substituting into the PDE:

$$XT'' = 4X''T \quad \Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{4T(t)} = \text{ constant } = -\lambda.$$

(Since x and t are independent, a function of x = function of t implies both must be constant.)

Thus we have two ODES: $X'' + \lambda X = 0$, $T'' + 4\lambda T = 0$.

As always, we have cases.

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$, $T(t) = c\cos(2\sqrt{\lambda}\,t) + d\sin(2\sqrt{\lambda}\,t)$.

Case (ii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c + dt$

Case (iii) $\lambda < 0$: Ignore –only gives trivial modal solutions.

Step 2. (Modal solutions) Find the separated solutions which also satisfy the BC.

For separated solutions, the BC are

$$X(0) = 0, \quad X(1) = 0.$$

Case (i) BC: $X(0) = a = 0$, $X(1) = a\sin(\sqrt{\lambda}) + b\sin(\sqrt{\lambda}) = 0$.

The nontrivial solutions have $a = 0$, b arbitrary and $\sin(\sqrt{\lambda}) = 0$. This implies $\sqrt{\lambda} = n\pi$, where $n = 1, 2, \ldots$.

Index solutions by n: $X_n(x) = \sin(n\pi x)$; $T_n(t) = c_n\cos(2n\pi t) + d_n\sin(2n\pi t)$; So,

$$y_n(x,t) = X_n(x)T_n(t) = \sin(n\pi x)(c_n\cos(2n\pi t) + d_n\sin(2n\pi t)).$$

Case (ii) BC: $X(0) = a = 0$, $X(1) = a + b = 0$.

Thus, $a = 0$, $b = 0$, i.e., there are only trivial solutions in this case.

Case (iii) We know this case only produces trivial solutions, so we skip it.

Step 3. Use superposition to give the general solution.

$$y(x,t) = \sum y_n(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)(c_n\cos(2n\pi t) + d_n\sin(2n\pi t)).$$

Step 4. Use the initial conditions to compute the coefficients.

$y(x,0) = \displaystyle\sum_{n=1}^{\infty} c_n\sin(n\pi x) = 0$. So $c_n = 0$ for all n.

$y_t(x,0) = \displaystyle\sum_{n=1}^{\infty} d_n 2n\pi\sin(n\pi x) = f(x)$. So $d_n 2n\pi$ are the Fourier sine coefficients of the function $f(x)$ on $[0,1]$. That is,

$$d_n 2n\pi = 2\int_0^1 f(x)\sin(n\pi x)\,dx \quad \text{or} \quad \boxed{d_n = \frac{1}{n\pi}\int_0^1 f(x)\sin(n\pi x)\,dx}.$$

With d_n as just defined, the Fourier solution to the problem is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} d_n\sin(n\pi x)\sin(2n\pi t)}.$$

(b) *Give the explicit solution to the equation of Part (a) when the initial velocity is given by $f(x) = x$ on $0 < x < 1$ (as if that were possible!).*

Solution: From the solution to Part (a) and the integral table we have

$$\int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^{n+1}}{n\pi} \qquad \Rightarrow\ d_n = \frac{1}{n\pi}\int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^{n+1}}{n^2\pi^2}.$$

Thus, $\boxed{y(x,t) = \sum_{n=1}^{\infty} d_n \sin(n\pi x)\sin(2n\pi t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2}\sin(n\pi x)\sin(2n\pi t)}$.

Problem 10. *Find the general real-valued solution to the system of DEs:*

$$x' = x - 2y, \quad y' = 4x + 3y.$$

Solution: Matrix equation: $\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}\mathbf{x}$.

Characteristic equation: $P(\lambda) = \lambda^2 - 4\lambda + 11 = 0$.

Eigenvalues: $\lambda = 2 \pm \sqrt{7}i$.

Basic eigenvector for $\lambda = 2 + \sqrt{7}i$: $\quad A - \lambda I = \begin{bmatrix} -1-\sqrt{7}i & -2 \\ 4 & 1-\sqrt{7}i \end{bmatrix}$. Can take $\mathbf{v} = \begin{bmatrix} -2 \\ 1+\sqrt{7}i \end{bmatrix}$.

A complex solution

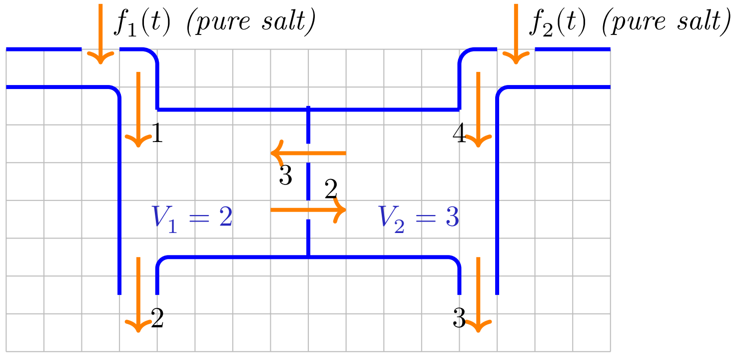
$$\begin{aligned}
\mathbf{z}(t) &= e^{(2+\sqrt{7}i)t}\begin{bmatrix} -2 \\ 1+\sqrt{7}i \end{bmatrix} \\
&= e^{2t}(\cos(\sqrt{7}t) + i\sin(\sqrt{7}t))\begin{bmatrix} -2 \\ 1+i\sqrt{7} \end{bmatrix} \\
&= e^{2t}\begin{bmatrix} -2\cos(\sqrt{7}t) - 2i\sin(\sqrt{7}t) \\ \cos(\sqrt{7}t) - \sqrt{7}\sin(\sqrt{7}t) + i(\sin(\sqrt{7}t) + \sqrt{7}\cos(\sqrt{7}t)) \end{bmatrix}
\end{aligned}$$

Both the real and imaginary parts of $\mathbf{z}$ are solutions to the system:

$$\begin{aligned}
\mathbf{x_1}(t) &= \mathrm{Re}(\mathbf{z}) = e^{2t}\begin{bmatrix} -2\cos(\sqrt{7}t) \\ \cos(\sqrt{7}t) - \sqrt{7}\sin(\sqrt{7}t) \end{bmatrix} \\
\mathbf{x_2}(t) &= \mathrm{Im}(\mathbf{z}) = e^{2t}\begin{bmatrix} -2\sin(\sqrt{7}t) \\ \sin(\sqrt{7}t) + \sqrt{7}\cos(\sqrt{7}t) \end{bmatrix}
\end{aligned}$$

General solution: $(x)(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$.

Problem 11. *Given the following two-tank mixing system with flow rates, inputs and volumes as shown. (All unit are compatible; $f_1(t)$ and $f_2(t)$ denote salt rates in $\frac{mass}{time}$.)*



(a) *Let x and y be the amount of salt in tanks 1 and 2 respectively. Set up a system of DEs modeling x, y.*

Solution: The tank is balanced (the volumes don't change) so,

$$x' = \text{rate in} - \text{rate out} = \left(f_1(t) + 3\cdot\frac{y}{V_2}\right) - \left(4\cdot\frac{x}{V_1}\right) \qquad = -2x + y + f_1$$
$$y' = \text{rate in} - \text{rate out} = \left(f_2(t) + 2\cdot\frac{x}{V_1}\right) - \left(6\cdot\frac{y}{V_2}\right) \qquad = x - 2y + f_2$$

(b) *Suppose the input salt rates $f_1(t)$ and $f_2(t)$ are constant. Show that the system approaches a state in which the final concentrations are constant.*

Solution: Our system is $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

Characteristic equation: $P(\lambda) = \lambda^2 + 4\lambda + 3 = 0$.

Eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = -3$.

Eigenvectors: For $\lambda_1 : \mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda_2 : \mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

General homogeneous solution: $\mathbf{x_h}(t) = c_1 e^{-t}\mathbf{v_1} + c_2 e^{-3t}\mathbf{v_2}$.

If $\mathbf{f} = (f_1, f_2)^T$ = constant then by guessing $\mathbf{x_p}$ = constant and plugging into $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ we get $\mathbf{0} = A\mathbf{x_p} + \mathbf{f} \Rightarrow \mathbf{x_p} = -A^{-1}\mathbf{f}$. That is,

$$\mathbf{x_p}(t) = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2f_1 + f_2 \\ f_1 + 2f_2 \end{bmatrix}.$$

The homogeneous solution $\mathbf{x_h(t)}$ decays to $\mathbf{0}$ as $t \to \infty$. Therefore, every solution goes asymtotically to the constant solution $\mathbf{x_p}$ as t gets large. This shows that the amount of salt in each tank becomes asymtotically constant. Since the volumes of fluid in each tank stays constant the concentrations must also become asymtotically constant.

For more problems on linear and nonlinear systems see Practice Quiz 7, Problems 8-11.

End practice final solutions